

Spacetime Diagrams

1 Getting Started

Now that we have the Lorentz transformations, we can convert an event's spacetime coordinates (x, y, z, t) in one frame to the same event's coordinates in another inertial reference frame, (x', y', z', t') . The equations are straight-forward but somewhat abstract. The spacetime diagram is a tool to help us visualize Lorentz transformations and multiple events in different reference frames.

2 One Reference Frame

In PHYS211, we often visualized motion in one dimension by plotting the x coordinate on the vertical axis and the t coordinate on the horizontal axis. An object moving at the speed of light would be represented by Figure 1. The slope of such a graph is the velocity of the object in the x -direction. For the speed of light, that is 3.00×10^8 m/s. You can confirm that this is the slope by examining rise over run.

In special relativity, we will flip that convention. We want to visualize time on the vertical axis, and position on the horizontal axis. If we simply flip the axes of Figure 1, we get something like Figure 2. Now, the slope of something moving at the speed of light is $\frac{1}{3.00 \times 10^8 \text{ m/s}}$. However, the speed of light is going to come up very often as we work in special relativity. More importantly, all observers agree on the speed of light. We will make our lives much easier if we design our diagrams such that the speed of light has a slope of 1, instead of dealing with more complicated units.

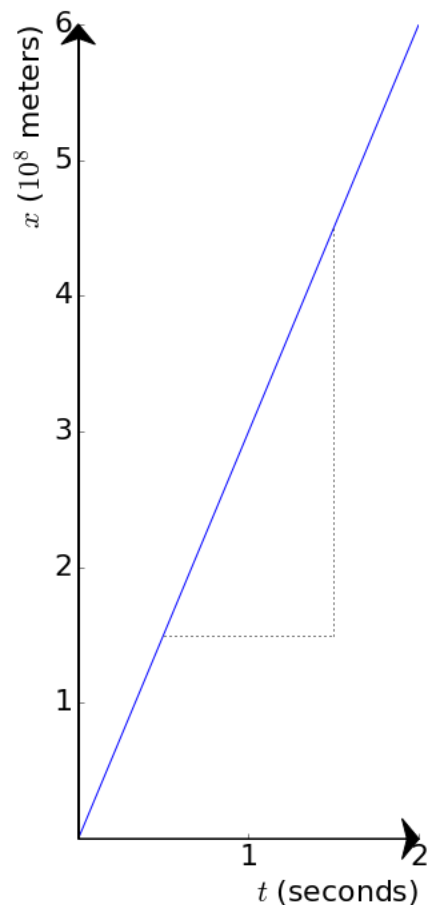


Figure 1: Traditional view of x -position vs. time of object moving at the speed of light.

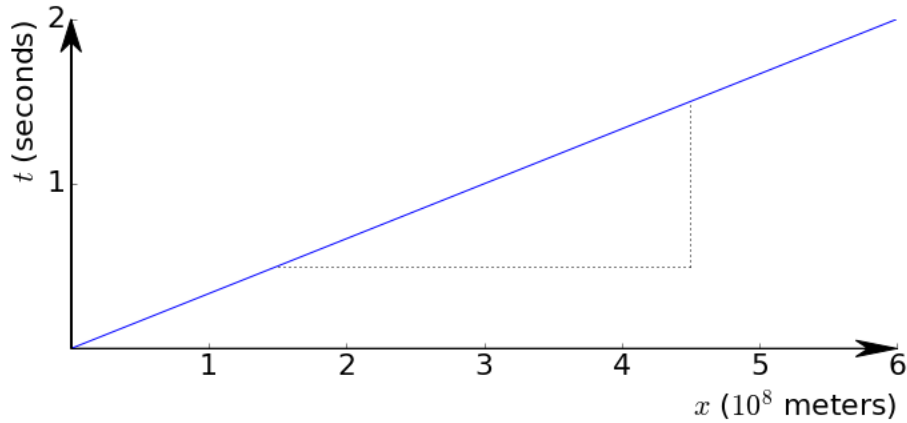


Figure 2: Flipped view, time vs. x -position of object moving at the speed of light.

2.1 "SR" Units

We have two options to simplify our units. The first option is to change how we plot position. Instead of using meters, let's use light-seconds. This is the distance that light travels in one second. We could choose any pair of units that keeps the slope 1; for example, minutes and light-minutes, years and light-years, etc. This is illustrated in the left side of Figure 3. In class, this is the convention that we will use. You could also change how you plot the time axis, instead. By multiplying by the speed of light, c , your time axis could instead be written in meters. This is illustrated in the right side of Figure 3.

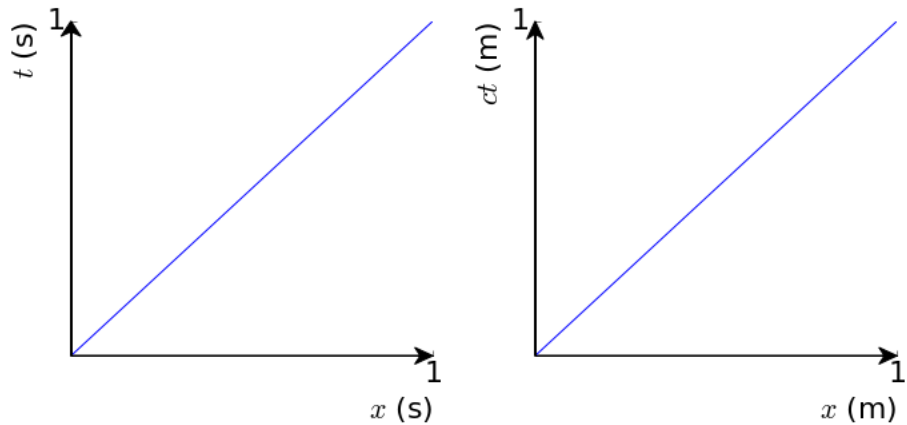


Figure 3: Two options for convenient units to make the speed of light equivalent to a slope of 1. Note that these two graphs *do not* represent the same amount of time that has passed. Which one represents a larger amount of time shown on the graph?

In ordinary SI units, the speed of light c is equal to 299,792,458 m/s, or to 2 significant figures, 3.00×10^8 m/s. As mentioned, this is an ungainly quantity. In the *Système International* (SI) system, the units for mechanical quantities (such as velocity, momentum, force, energy, angular momentum, and pressure) are based on the three fundamental units: the meter, the second, and the kilogram. When drawing spacetime diagrams, however, we will use a slightly modified version of SI units (let's call it *Système Relativistique*, or SR, units), where distance is measured in *seconds* (i.e. light seconds) instead of in meters (with the other basic units being the same). Note that by defining both distance and time in units of seconds, the quantity of velocity or speed becomes unitless.

The factors used to convert from SI distance units to SR distance units are based on the fundamental definition of the light-second: 3.00×10^8 m \equiv 1 s. For example, a distance of 25 km can be converted to distance in light-seconds as follows:

$$25 \text{ km} = 25 \text{ km} \times 1 \times 1 = 25 \text{ km} \times \frac{1000 \text{ m}}{1 \text{ km}} \times \left(\frac{1 \text{ s}}{3.00 \times 10^8 \text{ m}} \right) \quad (1)$$

$$= 8.3 \times 10^{-5} \text{ s} = 83 \mu\text{s} \quad (2)$$

This means that 25 km is equivalent to the distance that light travels in 83 millionths of a second. For large distances, it may be more appropriate to measure distances in hours or years instead. Can you show in your Reading Memo that a lightyear is equivalent to 9.5×10^{15} meters?

2.2 Worldlines

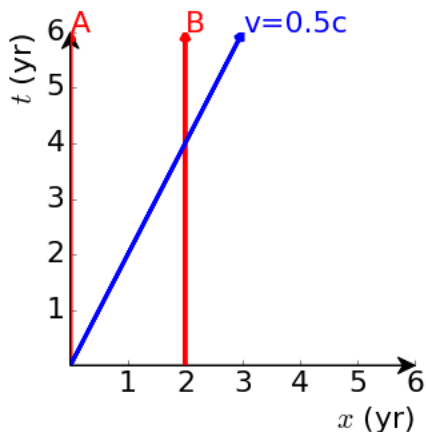


Figure 4: A space-time diagram of two planets (A and B) stationary to one another, and a spaceship traveling at $0.5c$ from A to B.

Imagine the following scenario: Planet A is at rest at the origin ($x = 0$) of their inertial frame of reference. Planet B is also at rest relative to Planet A, but at a distance 2 light-years away. In Figure 4, this is represented by two red lines labeled “A” and “B”. These are the **worldlines** of the respective planets. Their location remains the same as time goes on. When we say “as time goes on,” we imply that we read the spacetime diagram from bottom ($t=0$) to top. Imagine Planet A sends a spacecraft to Planet B at a speed of one half the speed of light ($0.5c$). The worldline of this spacecraft is represented by the blue line. It is sloped, because its position changes with time. The slope on this graph is 2. For every 2 years traveled (rise), the spacecraft travels a distance of 1 light-year (run).

How long does the spacecraft take to get to Planet B? From the definition of velocity alone, you can find $t = d/(0.5c) = 2 \text{ light-years} / (0.5 \times 1 \text{ light-year/year}) = 4 \text{ light-years}$. Notice that this is also the time coordinate where the blue worldline of the spaceship coincides with the red worldline of Planet B in Figure 4. The coordinates are $(x,t) = (2 \text{ light-years}, 4 \text{ years})$. From now on, instead of saying "light-years", let's just stick to SR units and simply call this distance "years."

3 Two-Observer Spacetime Diagram

In the previous example, we only looked at one inertial reference frame, the one in which Planets A and B are at rest. But what about the reference frame of the spaceship? We know that at such a high speed, the spaceship experiences space and time differently than the two planets do. The spaceship does not believe it takes 4 years to travel from A to B. How can we represent that on *one* graph? The key is to recognize that **in the spaceship's frame of reference, the spaceship is at rest, and so its worldline would be parallel to the time axis in its frame of reference, t'** . Therefore, the blue line in Figure 4 is in fact one of the axes for the spacetime diagram of the spaceship's frame of reference.

Before we continue, here are a few conventions we will follow:

- We will work with only those events that occur along the common x and x' axes of the two frames. Assume $y' = y = z' = z = 0$ for all events under discussion.
- The spatial origins ($x = x' = 0$) of both frames coincide with $t = t' = 0$, which we will call the Origin Event.
- We must pick one frame S to designate as the Home Frame. It is convention to pick the Other Frame (S') to be the frame of the two that moves in the $+x$ direction with respect to the Home Frame.
- We will represent the Home Frame t and x axes in the *usual* manner in a spacetime diagram (i.e., we will draw its t axis as a vertical line and its x axis as a horizontal line).
- Relativistic effects are not obvious at speeds much less than c , so represent all velocities as $\beta = v/c$.
- All observers agree on the speed of light, so the speed of light should be a line with slope=1 in *both* coordinate systems.

In our example, the origin event is when the spaceship left Planet A. The spaceship and Planet A both coincided at the same location $x = x' = 0$ and they both declared that moment to be $t = t' = 0$.

We've already identified that the blue line in Figure 4 represents the t' axis. It connects all events that occur at the spatial origin $x = 0$ of that frame (i.e. at the same *place* as the origin event - in this case, the location of the spaceship in its own frame). It is a line with a slope of $1/\beta$. Where is the x' axis? This would be a line connecting all events that occur at $t = 0$ in that frame (i.e., at the same *time* as the origin event). To "give away the punchline," the x' axis will be a line of slope β . *The diagram x' axis makes the same angle, θ , with the diagram x axis that the t' axis makes with the t axis..* This is illustrated in Figure 5. This assures that a beam of light will still have slope = 1 in the moving frame. We will confirm this later.

3.1 Calibrating the axes

Where do we draw the tick marks for the t' axis - in other words, how do we mark where $t' = 1$ year, 2 years, etc.?

One way to figure this out is to use the time dilation equation.

$$\Delta t = \frac{1}{\sqrt{1-\beta^2}} \Delta t' \quad (3)$$

The quantity $\frac{1}{\sqrt{1-\beta^2}}$ appears so often, it is given the symbol γ ("gamma"). So, $\Delta t = \gamma \Delta t'$. This means that $t' = (0, 1, 2, 3\dots)$ when $t = (\gamma \times 0, \gamma \times 1, \gamma \times 2, \gamma \times 3\dots)$. In the case of this example, where $\beta = 0.5$, $\gamma \approx 1.15$.

Let's think of this in terms of events. An event appears in one singular place on our graph. The spaceship's location after 1.15 years have passed on Planet A is represented by a gray star in Figure 6. This corresponds to 1 year in the spaceship's reference frame, so we can draw a tickmark and label it $t' = 1$ year. Likewise, at every interval spaced every 1.15 years in the Home Frame t axis, we can draw a line straight across to the t' axis and know that 1 more year has passed for the spaceship in the Other Frame. These intervals are shown in dashed lines on Figure 6.

Let's practice reading the graph. We'll label the event when the spaceship reaches Planet B with a blue star. This occurs at $(x, t) = (2 \text{ light-years}, 4 \text{ years})$. On the t' axis, however, this corresponds to what looks like a little less than 3.5 years, just reading by eye. Let's check using the time dilation equation:

$$\Delta t' = \sqrt{1-\beta^2} \Delta t = \sqrt{1-0.5^2} (4 \text{ yr}) = 3.46 \text{ yr} \quad (4)$$

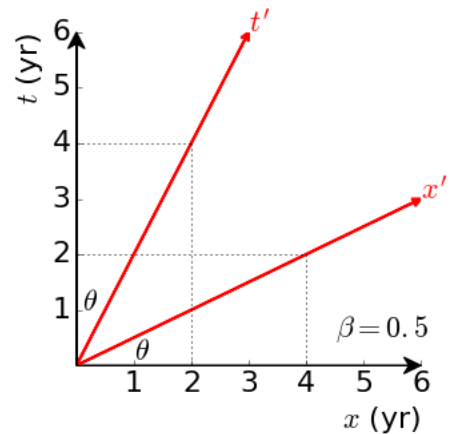


Figure 5: Spacetime diagram for an Other Frame S' moving at $\beta = 0.5$.

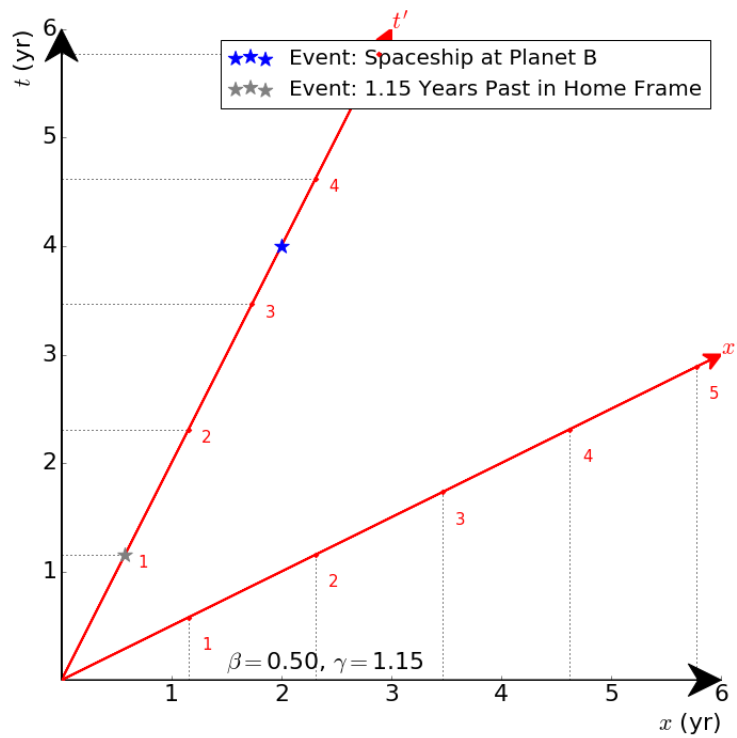


Figure 6: Dotted lines illustrate calibration of the t' and x' coordinate axes. Two events are also shown; see text for more explanation.

We came to the same conclusion by reading the graph that we can also come to using the time dilation equation.

The x' axis can be calibrated in the same way by drawing lines straight up from the x -axis. This is also shown in Figure 6.

3.2 Calibrating the axes using spacetime intervals

We used a very special case to calibrate our axes above. There is a more general way to do this using the invariant spacetime interval and the Lorentz transformations. Consider two events located at (x_1, t_1) and (x_2, t_2) in the Home Frame. Define an interval between them as $\sqrt{(x_2 - x_1)^2 - (t_2 - t_1)^2} = \sqrt{\Delta x^2 - \Delta t^2}$. This is analogous to the Euclidean distance between two points, but with the minus sign in front of the time difference.

Let's see how this quantity looks in the moving frame. For this, we cannot simply use the time dilation and Lorentz contractions because simultaneous events in one frame are not simultaneous in another. As an example, consider observing the simultaneous explosions of two stars going supernova. In one frame, the two stars are at rest and we only have a time difference to measure. In a frame moving at β with respect to this one, the two explosions are at different times as well as distances, so we have a length difference and a time difference to consider. No only that, but because of the nonsimultaneity, the locations are measured at different times and we must resort to the full Lorentz transformation to calculate the interval in the moving frame.

The Lorentz and inverse Lorentz transformations, using $\gamma = 1/(\sqrt{1 - \beta^2})$, are:

$$x = \gamma(\beta t' + x') \quad (5)$$

$$t = \gamma(t' + \beta x') \quad (6)$$

$$x' = \gamma(-\beta t + x) \quad (7)$$

$$t' = \gamma(t - \beta x) \quad (8)$$

At this moment you might be thinking: "Stop! Professor, you have forgotten to multiply each factor of time by c !" Let's stick with SR units, where c is a unitless quantity equal to 1. This means we don't need to write it (but it wouldn't be incorrect if you did).

By taking the difference $x_2 - x_1$ and $t_2 - t_1$, etc., you can show that:

$$\Delta x = \gamma(\beta \Delta t' + \Delta x') \quad (9)$$

$$\Delta t = \gamma(\Delta t' + \beta \Delta x') \quad (10)$$

$$\Delta x' = \gamma(-\beta \Delta t + \Delta x) \quad (11)$$

$$\Delta t' = \gamma(\Delta t - \beta \Delta x) \quad (12)$$

These are just like the Lorentz and inverse Lorentz transformations, except it replaces the coordinate quantities with the corresponding coordinate differences. Let's use the last two equations above to solve for the interval defined earlier in this section.

$$(\Delta x')^2 - (\Delta t')^2 = (\gamma(-\beta\Delta t + \Delta x))^2 - (\gamma(\Delta t - \beta\Delta x))^2 \quad (13)$$

$$= \gamma^2(+\beta^2(\Delta t)^2 - \beta\Delta t\Delta x + (\Delta x)^2) - ((\Delta t)^2 - \beta^2\Delta t\Delta x + \beta^2(\Delta x)^2) \quad (14)$$

$$= \gamma^2(+\beta^2(\Delta t)^2 + (\Delta x)^2 - (\Delta t)^2 - \beta^2(\Delta x)^2) \quad (15)$$

$$= \gamma^2((1 - \beta^2)(\Delta x)^2 - (1 - \beta^2)(\Delta t)^2) \quad (16)$$

$$= \frac{1}{1 - \beta^2}((1 - \beta^2)(\Delta x)^2 - (1 - \beta^2)(\Delta t)^2) \quad (17)$$

$$(\Delta x')^2 - (\Delta t')^2 = (\Delta x)^2 - (\Delta t)^2 \quad (18)$$

We've found the invariant spacetime interval. If we take Δx as the difference between x and the origin and Δt to be the difference between t and the origin, then the equation for a hyperbola passing through the point $x = 0, t = t$ will be:

$$(\Delta t)^2 - (\Delta x)^2 = t^2 \quad (19)$$

For points on the x -axis, we similarly get:

$$(\Delta x)^2 - (\Delta t)^2 = x^2 \quad (20)$$

We can draw these hyperbolae onto our coordinate grid, as shown in Figure 7 as dashed gray lines. Then, draw the coordinate axes (red lines) for the home frame, and the places where the coordinate lines intersect with the hyperbolas are the grid points. Finally, how do you read the correct t' and x' coordinates for an event that doesn't take place right on one of the axes? The grid lines for the Other Frame are drawn parallel to the axes. This is shown in the right hand side of Figure 7.

A summary of the steps for drawing and interpreting a two-observer diagram are shown in Figure 8

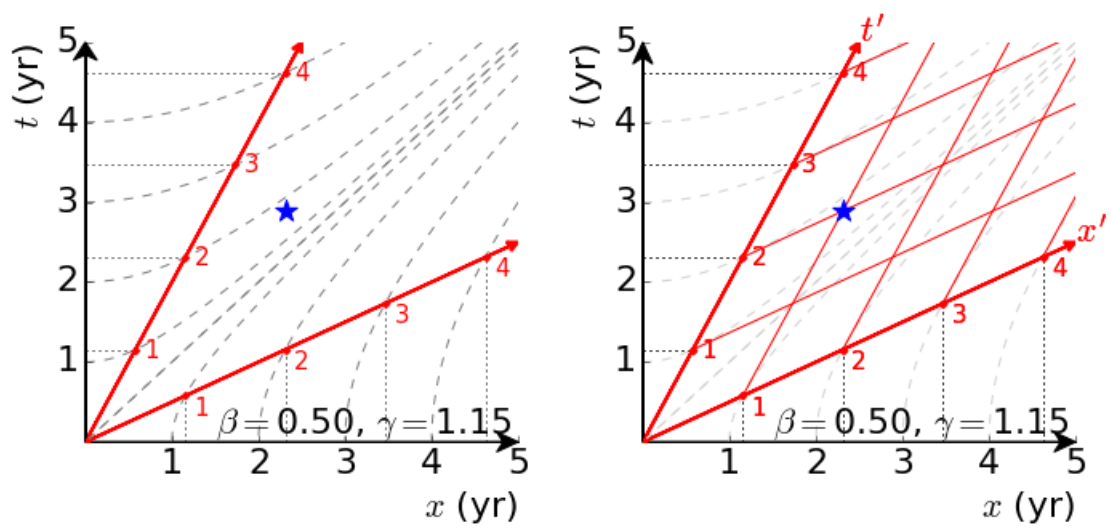
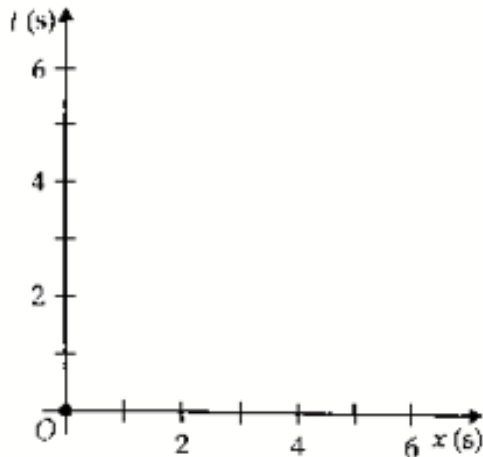
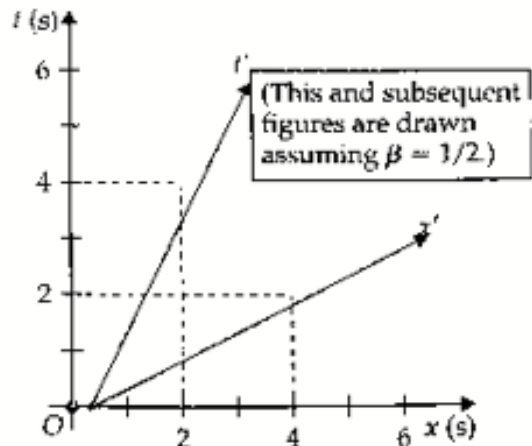


Figure 7: Dashed gray lines show hyperbolae used to calibrate the coordinate axes for t' and x' . On the right, grid lines for these axes are shown in red. In the figure, an event is shown to appear at $(x', t') = (1, 2)$ yr. This correspond to $(x, t) = (2.3, 2.9)$ yr.



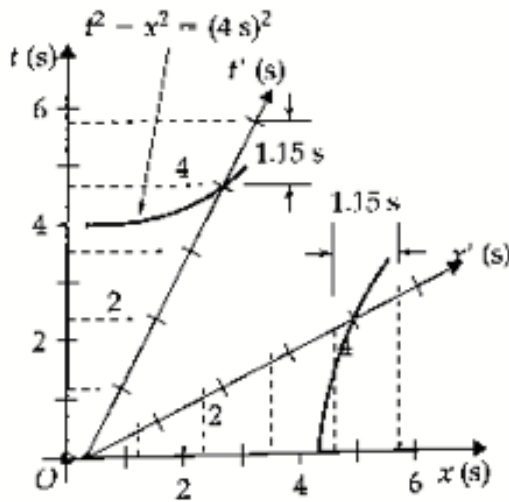
Choose one frame to be the Home Frame. Draw its axes in the usual manner, and indicate the position of the origin event, O . Calibrate the axes with some convenient scale.

(a)



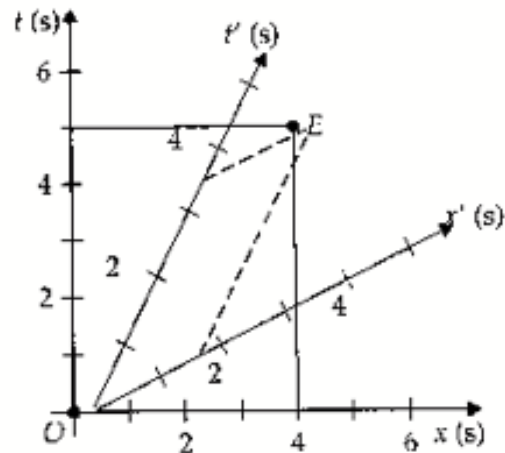
Draw the t' axis of the Other Frame from the origin event O with a slope $1/\beta$ (where β is the x -velocity of the Other Frame with respect to the Home Frame). Draw the diagram x' axis of the Other Frame with slope β .

(b)



Calibrate the t' axis using hyperbolas or by drawing marks that are vertically separated by $\Delta t = \gamma \Delta t'$, where $\Delta t'$ is the intended mark separation in the Other Frame (1 s in this case). Calibrate the diagram x' axis with marks having the same spacing. (When $\beta = 1/2$, $\gamma = 1.15$.)

(c)



Read the coordinates (in either frame) of any event E by dropping parallels from the event to the appropriate axes. (In this case, E has the coordinates $t = 5$ s, $x = 4$ s in the Home Frame and $t' = 3.4$ s, $x' = 1.7$ s in the Other Frame.)

(d)

Figure 8: Summary of the steps in drawing and interpreting a two-observer diagram. Figure R6.1 of Moore.

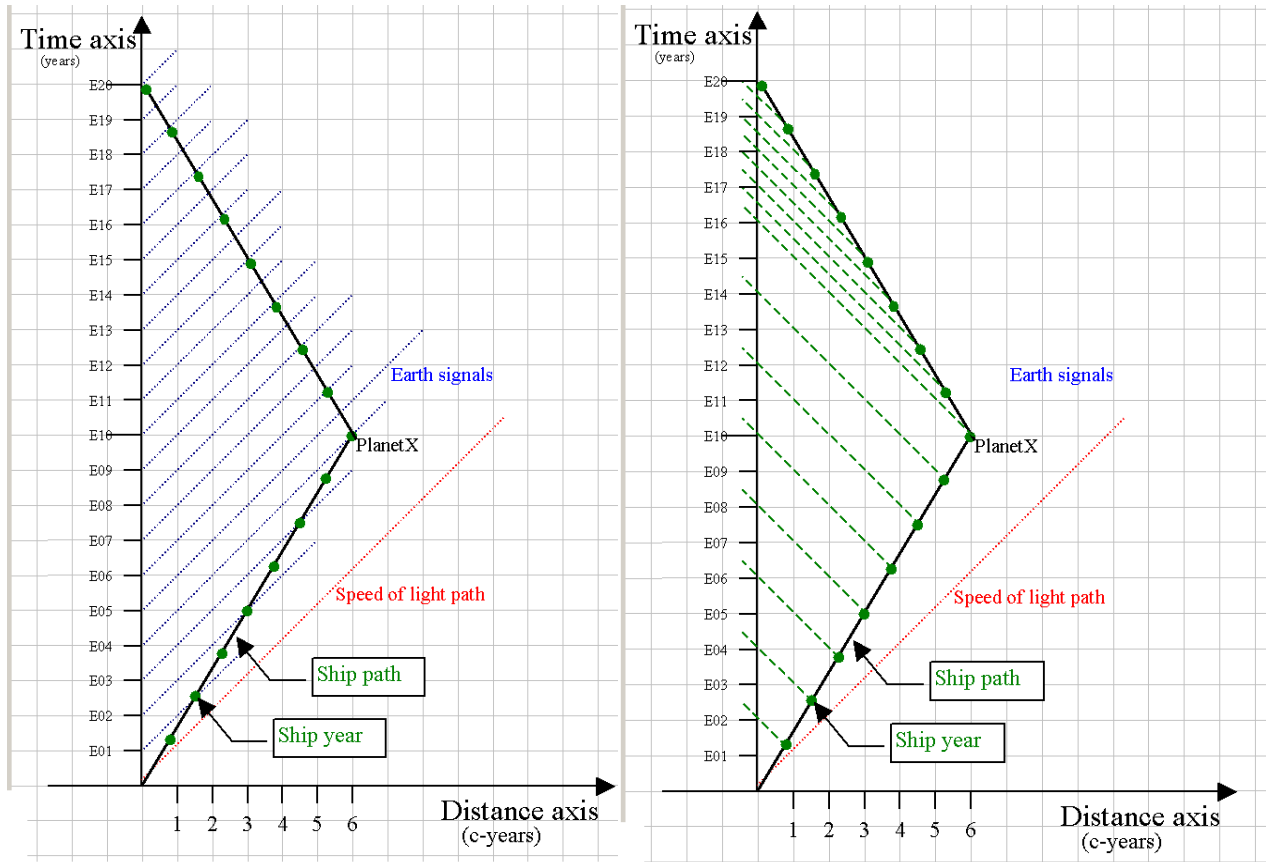


Figure 9: The Twin "Paradox", images from S. Schneider.

4 Using Spacetime Diagrams: The Twin Paradox

Two identical twins start at a common place and time, at the origin of coordinates. One of the two then leaves on a trip at $\beta = 0.6$, travels out a distance of 6 light-years to "Planet X" and then returns at $\beta = -0.6$. It will take 10 Earth years to get to the planet, and 10 years back (total time of 20 years elapsed on Earth). But, at that speed, the time dilation equations tell us that the ship will experience 8 years of time elapsed on the way out, and 8 years on the way back ... so 16 ship years, for 20 earth years. Imagine both twins send a signal to the other twin every time a year has passed.

First, set up the diagram with the two sets of axes. Include the return trip with the opposite slope. Let's count how many signals the ship receives from the Earth (notice the blue Earth signal lines are moving upward diagonally, spaced every 1 Earth year, left of Figure 9). Also, the green dots on the ship path are the ship year markers.

On the way out - for every 2 ship years, it gets one signal (this fits the relativistic Doppler shift for this speed - should be 1/2 signal rate - not discussed yet). As soon as the ship turns around and heads back (now the ship path heads back toward the time axis) - the

rate doubles to 2 earth signals per ship year. So, on the way out - $(8)(1/2) = 4$ Earth signals ... on the way back $(8)(2) = 16$ signals ... total = 20 Earth signals, as expected.

What if we turned it around, and looked at the ship sending out signals to the Earth. In this case, when the ship thinks a year has passed, it sends a signal to the Earth (green lines of unit slope - heading back toward the time). Notice when the ship heads toward the planet, the Earth receives that 1/2 rate. At the Earth's 10th year, the ship has technically arrived at the planet, but the signal has just left the planet, and will take 6 years to get back ... notice that for the Earth, the signal rate doesn't change until the "arrival signal" has gotten there. Only then does the rate double. So, to add up the time, $(16)(1/2) = 8$ signals until arrival - then $(4)(2) = 8$ signals thus 16 ship signals received by the Earth.

Notice the asymmetry in the problem due to the ship's acceleration in the middle. This is what fixes the difference in ages of the twins; the one who accelerated is the younger one. Velocity is relative, acceleration is not. The supposed "paradox" is both twins would expect the other to be younger; this is resolved by recognizing that the moving twin is not in an inertial frame of reference for the trip.

This document was prepared based on Six Ideas That Shaped Physics by Thomas A. Moore, 2nd Edition, Unit R: The Laws of Physics are Frame-Independent, Scott Schneider's Contemporary Physics course at Lawrence Technological University (Twin Paradox), and includes direct quotations. Spacetime diagrams were created in Python.