Recall from our last class, for probability density $\rho(x)$

- $P[a \leq x \leq b] = \int_a^b \rho(x)\,dx =$ Probability of finding particle between *a* and *b*
- $\langle x \rangle = \int_{-\infty}^{+\infty} x\rho(x)\,dx =$ Average or "Expectation Value" of *x*
- $\int_{-\infty}^{+\infty} \rho(x)\,dx = 1$ [Normalization condition]

Part I: A Simple Wavefunction
Imagine a ball is constrained to move along a line inside of tube of length 2*L*, centered about position 0. The ball is equally likely to be found anywhere in the tube at some time *t*.

1. Sketch the probability density $|\psi(x)|^2$ of the ball as a function of position. Don't forget to label and scale your axes in this and other questions, and normalize!

2. Sketch the wavefunction $\psi(x)$ of the ball as a function of position.

3. Can you think of *another* wavefunction $\psi(x)$ that would also satisfy your probability density from Question 1? If so, draw it. If not, explain why not.

4. What is the probability of finding the ball in the left half of the tube? (You may intuit it is 50%, but show this using the probabilistic interpretation of the wave function.)

5. What is the expectation value (average value) of the position? (You may intuit it is 0, but show this using the probabilistic interpretation of the wave function.)

Part II: Review of Complex Numbers

A complex function is one that contains one or more imaginary numbers, $i = \sqrt{-1}$. The complex conjugate of a function is obtained by replacing every occurrence of i in that function with $-i$.

1. If $a = 3 + 4i$, what is the complex conjugate of a, which we denote as a^*?

   $a^* =$

2. What is the product of a^* times a?

3. The probability density can be thought of as the complex conjugate of the wavefunction times the wavefunction:

$$\rho(x)dx = |\psi(x)|^2 dx = \psi^*(x)\psi(x)dx$$

   Is the product of an imaginary function and its complex conjugate always real, always imaginary, or does it depend on something else?

4. Can a wavefunction be imaginary? Justify why or why not.

Part III: Wavefunction Normalization

For a wavefunction to be properly normalized, $\int_{-\infty}^{+\infty} \rho(x)\,dx = \int_{-\infty}^{+\infty} \left|\psi(x)\right|^2 dx = 1$

Suppose that a particle with energy *E* is moving along the *x*-axis and is confined in the region between 0 and *L*. One possible wave function is:

$$\psi(x) = \begin{cases} Ae^{-\frac{iEt}{\hbar}} \sin\left(\frac{\pi x}{L}\right) & \text{, when } 0 \leq x \leq L \\ 0 & \text{, otherwise} \end{cases}$$

This wave function depends on time as well as position, and is complex! This allows the wavefunction to oscillate with time. Don't worry; this won't be a problem in the steps you are about to take. Though we don't yet know *why* the wavefunction is complex, I want us to see that the math still works out.

1. What is the probability density? What does it depend on?

2. Determine the normalization constant, A. **Hint:** Write an equation that satisfies the normalization condition and solve for A. You may need to look up a trigonometric identity to reduce the power of a trigonometric function.

3. Sketch the approximate probability density, where your horizontal axis is position in multiples of *L, i.e.* (*x*/*L*), and your vertical axis is $\left|\psi(x)\right|^2 \big/ L$.

4. In part 5, you'll be asked to find the probability of finding the particle from 0 to *L*/3. First, use your sketch to make a prediction: approximately what probability do you expect? Will it be more or less than 1/3? Explain.

5. What is the probability of finding the particle from 0 to *L*/3?

Part IV: Using Mathematica for Visualization

Mathematica offers many powerful tools for visualizing wavefunctions and probability densities, as well as doing the many calculations you just did by hand. Open the Mathematica notebook titled *phys301-wavefunction.nb* and follow the instructions. You'll need to enter some of your own formulas that you found by hand.

Part V: Using Mathematica for Calculation

Use the example commands to confirm that your wavefunction is properly normalized, check your probability for finding the particle from 0 to *L*/3, and calculate the expectation value.

Part VI: Using Mathematica to Visualize the Wavefunction

Because the wavefunction is imaginary, we cannot simply plot it as a function of position for a specified time. We can, however, visualize the real and imaginary parts, separately.

1. First, use Euler's formula $e^{ix} = \cos x + i \sin x$ to write the wavefunction in terms of a real part plus an imaginary part.

2. Discuss your formula. Does at least one part look familiar? It should! What do you expect to see when you animate this part? What about the other part? Make as concrete and specific of a prediction as you can before moving on.

3. Use the provided command in Mathematica to visualize the real and imaginary parts of the wavefunction as a function of space and time. Describe what you saw. How good was your prediction?

Parts I and III adapted from OpenStax University Physics Volume 3.