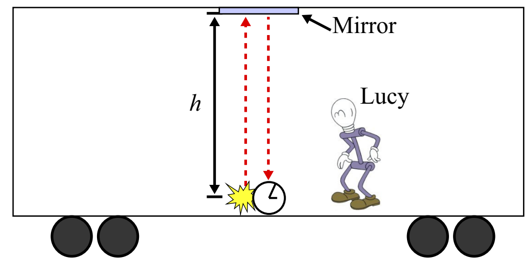
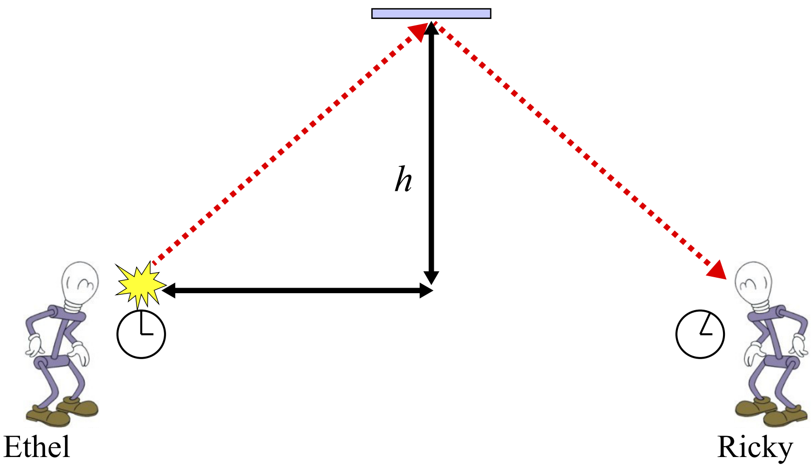
Derivation of Time Dilation Equation



In the rest frame of the time clock (pictured to the right), an observer measures that the light travels a distance of $2h$ at a speed of c. The time interval, which in this special case is the *proper time*, is $\Delta\tau = 2h/c$.

Below, we see the situation as measured by two observers in a frame moving relative to the light clock at speed v. Ethel and Ricky measure a different time interval, Δt, than Lucy. The steps below will walk you through the procedure of determining the relationship between Δt and $\Delta\tau$.



1. In the figure above, a triangle has been drawn, with one side labeled as distance h. Label the other two distances in terms of v, c, and Δt. Hints: (1) use the relationships between velocity, distance, and time. (2) Be careful, how much time has passed when the light beam hits the mirror, all of Δt or only a fraction?
2. Use the Pythagorean theorem to write an equation relating all three of your labeled distances.

3. Solve your equation for Δt. You should find it equal to a fraction.

4. Multiply both the numerator and denominator of your fraction by $1/c$.

5. Substitute the expression $\Delta\tau = 2h/c$ in for your numerator.

6. Simplify your denominator so that your final equation reads $\Delta t = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}}$.

7. Your equation includes a number of factors that are so common in Special Relativity, we assign them special symbols. Let $\beta = \frac{v}{c}$ ("beta") and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ ("gamma"). Further simplify your equation for Δt using these new expressions.

8. What is the minimum value that γ can be? Under what circumstances?

9. What is the maximum value that γ can be (or can approach)? Under what circumstances?

10. Is Δt always greater than $\Delta\tau$, always less than $\Delta\tau$, or does it depend on something? If so, what?