

Part I: The Quantum Oscillator Wavefunctions

For a (classical) oscillator potential of $U(x) = \frac{1}{2}kx^2 = \frac{1}{2}(m\omega^2)x^2$, the solutions to the Schrödinger equation are, for quantum number $n = 1, 2, 3$:

$$\psi_1(x) = A_1 e^{-x^2/(2b^2)}$$

$$\psi_2(x) = A_2 \frac{x}{b} e^{-x^2/(2b^2)}$$

$$\psi_3(x) = A_3 \left(1 - 2 \frac{x^2}{b^2} \right) e^{-x^2/(2b^2)}$$

The allowed energy levels are given by the formula: $E_n = \left(n - \frac{1}{2}\right)\hbar\omega$ for $n = 1, 2, 3, \dots$

1. The above energy level equations states that the energy levels of the harmonic oscillator are quantized. This is *not* a new idea; we encountered it in the development of quantum mechanics well before Schrödinger came onto the scene. Try to remember: what did the quantization of oscillator energy help *explain*? Do you remember *who* came up with it?

We'd like to visualize the *time-dependent* wave-functions in Mathematica. Before we can do that, we'll need to solve for the normalization constant.

2. Find A_1 . You may use the identity $\int_{-\infty}^{+\infty} e^{-y^2} dy = \sqrt{\pi}$.

3. Open the Mathematica notebook *phys301-quantumoscillator.nb*. Run Cell 1 using Shift+Enter, which sets all our constants to 1; we're just interested in the general shape of the wavefunctions. Run Cell 2, which defines all the equations we'll use. Write, in words, what equations are included in this cell.

4. Notice that after you run Cell 2, the variables A1, A2, and A3 are highlighted in blue. This is Mathematica telling us that they are undefined. In the blank space provided, input a formula for the constant A1 that you found above. Expressions for the other constants are already included. Run the cell.
5. Input three functions in Cell 4 to check that we have correctly normalized the first three wave functions. Describe what you did. Is the normalization correct?
6. Run Cell 5 to plot the first three wave functions, as well as the potential.
7. Modify the command in Cell 6 so that the wave function looks more like the image in your book, spaced out vertically according to energy level. Does the spacing between energy levels get larger, smaller, or stay the same as you go to higher energy levels?

Part II: The Correspondence Principle

1. A classical harmonic oscillator might be a mass on a spring (with zero friction or air resistance). We define the equilibrium position of the spring to be $x = 0$. Imagine you displace the spring a distance Δx and let it go. Some random time later, you come back and take a picture of the system. What is/are the most likely position(s) where the spring will be in your picture? What is/are the least likely position(s)?
2. Run Cell 7 to create an interactive plot of the probability density of the quantum harmonic oscillator for different levels of n . Does this system obey the correspondence principle? Why or why not?
3. Some additional practice with classical systems will help you on the next part. Imagine you have a spring with spring constant $k = 9.87 \frac{\text{N}}{\text{m}}$ and you attach an object of mass $m = 1.0 \text{ kg}$. For a classical oscillator, $U(x) = \frac{1}{2}kx^2$ and $\omega = \sqrt{\frac{k}{m}}$.

Part III: Time-Dependence

1. What is the time dependent portion of the wavefunction, $\phi(t)$? Input this formula into Cell 8. The other formulas define the full, time-dependent wavefunction, $\Psi_n(x,t) = \psi_n(x)\phi_n(t)$.
2. The wave function is a complex number which can be broken up into real and imaginary parts. We can visualize a complex number using the complex plane, with the real component on the horizontal axis and the imaginary component on the vertical axis. For the complex number $-2 + 3i$, draw its location on the complex plane, and find the absolute value (or modulus) of the complex number.
3. Run Cell 9 to create animations of the real and imaginary parts of the time-dependent wave function, with dashed lines. The absolute value (or modulus) of the wave function is shown as a solid line. Describe what you see.
4. Two students are debating what they see on the screen.
 Student 1: "The wave-function oscillates faster as we go to higher n . This means the period of the oscillating system is smaller, and therefore the angular frequency greater. In our classical analogy, this is like replacing the original spring with a stiffer spring every time we increase n ."
 Student 2: "I see what you're saying, but we also found the maximum speed of the stiff spring pulled to 10 cm was the same if we pulled the original spring 20 cm. So maybe as we are increasing n , we are really increasing the distance that the particle can travel - like if this were an infinite square well, we'd be increasing the length of the box."

Both students have errors in their reasoning. Explain what each student has wrong, and provide a correct explanation to why the wave function oscillates faster as we go to higher n , including a classical analogy. **Hint:** as all things quantum mechanical, think energy.

5. Though incorrect in their reasoning, Student 1 brings up an interesting question. What happens if we increase ω of our system? In the classical analogy, this is like using a stiffer spring. For each of the graphs, predict what changes you will see if you replace $\omega = 1$ with $\omega = 2$.

Cell/Graph	Prediction	Observation
Cell 5: Graph of potential energy (dashed line)		
Cell 5: Graph of first three wave-functions (solid lines)		
Cell 6: Graph of first three wave functions (solid lines), spaced out vertically according to energy level.		
Cell 9: Animations of real (dashed), imaginary (dashed) and absolute values (solid) of first three wavefunctions.		

6. When you've finished your predictions, replace $\omega = 1$ with $\omega = 2$ in Cell 1. Re-run the cells, in order, and record how your observations compared to your predictions.