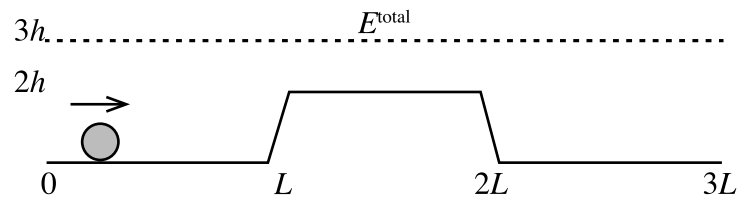
Quantum Tunneling

Part A: Classical Particle

A ball of mass m rolls to the right on a flat, frictionless surface with total energy $E = 3mgh$. The ball soon encounters a sloped surface and rolls up to height $2h$. After, the ball rolls back down the ramp, always staying in contact with the surface.



(a) Is the total energy of the system as the ball rolls from 0 to $3L$ increasing, decreasing, or staying the same?

(b) Sketch the kinetic energy, gravitational potential energy, and total energy of the system between 0 and $3L$.

(c) Is the amount of time the ball spends between L and $2L$ greater than, less than, or equal to the amount of time it spends between 0 and L? How does it compare to the amount of time it spends between $2L$ and $3L$? (Ignore the time the ball spends on the slope.)

(d) Now imagine that we take a photograph of the ball at some random time *before* it reaches $2L$. Is the probability of finding the ball between 0 and L greater than, less than or equal to the probability of finding it between L and $2L$? Why?

Part B: Classical particle with $E < V$

Now imagine that the same ball from PART A has an initial total energy of $E = mgh$, while the height of the hill remains at $2h$.

(a) What happens to the ball as it starts to go up the hill? Is it possible for the ball to be found between L and $2L$? How about between $2L$ and $3L$?

Part C: Solutions to Schrödinger's equation

The time-independent Schrödinger equation is given by:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)+V(x)\psi(x)=E^{total}\psi(x) \tag{1}$$

This can be rewritten as:

$$\begin{aligned}\frac{d^2\psi}{dx^2} &= -\frac{2m}{\hbar^2}\left(E-V\right)\psi \\ &= \frac{2m}{\hbar^2}\left(V-E\right)\psi\end{aligned} \tag{2}$$

(a) If $\boldsymbol{E} < \boldsymbol{V}$, will the solutions to Schrödinger's equation be real exponentials or complex exponentials? [Hint: Is the quantity on the right-hand side positive or negative in this case?]

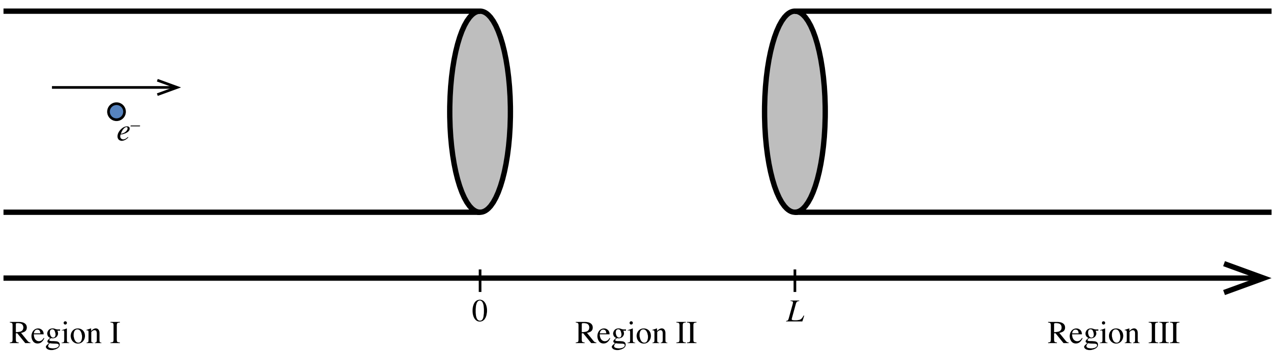
(b) Write down the most general solution to Schrödinger's equation for the case when $\boldsymbol{E} < \boldsymbol{V}$.

(c) If $\boldsymbol{E} > \boldsymbol{V}$, will the solutions to Schrödinger's equation be real exponentials or complex exponentials? [Again, consider whether the right-hand side is positive or negative.]

(d) Write down the most general solution to Schrödinger's equation for the case when $\boldsymbol{E} > \boldsymbol{V}$.

Part D: Electron in a wire ($E > V$)

Consider an electron with total energy $\boldsymbol{E}$ moving to the right through a very long smooth copper wire with a ***small*** air gap in the middle:



Assume that the work function of the wire is V_0 and that $V = 0$ inside the wire.

(a) If $E > V_0$, draw a graph of the system's potential energy in all three regions. Also draw a dashed line indicating the total energy of the system.

(b) In each of the three regions, are the solutions to Schrödinger's equation real exponentials or complex exponentials? Write down a solution for each of the three regions corresponding to an electron traveling to the right.

Region I:

Region II:

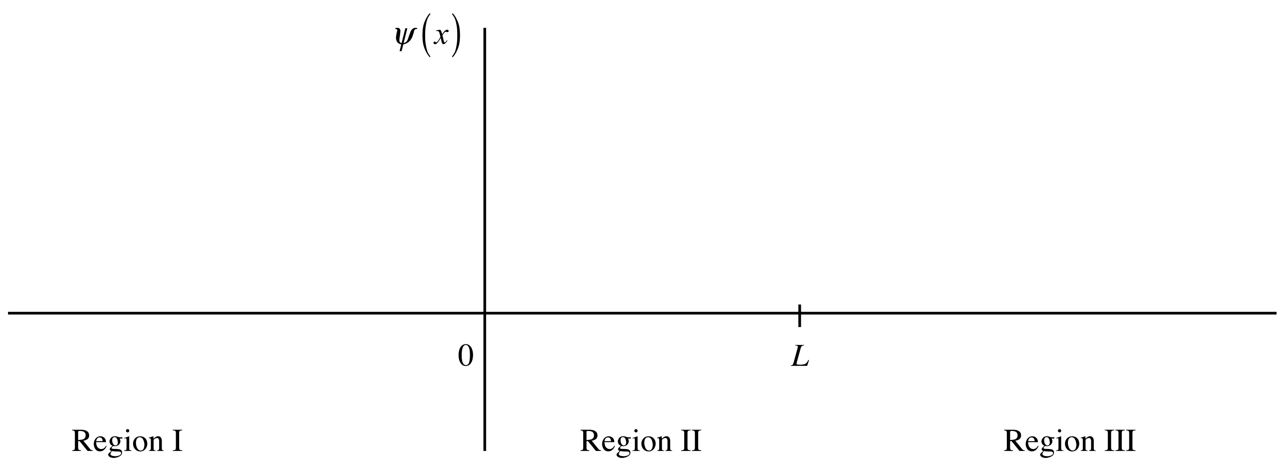
Region III:

(c) How does the kinetic energy of the system (electron) compare in each of the three regions? Rank the kinetic energies in the three regions (***KE*** $_1$, ***KE*** $_2$ & ***KE*** $_3$) from high to low.

(d) How does the deBroglie wavelength of the electron compare in each of the three regions? Rank the wavelengths in the three regions (λ_1, λ_2, λ_3) from largest to smallest. If the wavelength is not defined in a particular region, then say so.

(e) How does the amplitude of the electron's wave function compare in each of the three regions? [Hint: think about what $|\psi(x)|^2$ tells you in terms of probabilities].

(f) With this information in mind, sketch the ***real part*** of the electron's wave function in all three regions:



Part E: Electron in a wire ($E < V$)

Consider the same situation as in **Part D**, but now the total energy E of the electron is ***less than*** the work function V_0.

(a) If $E < V_0$, draw a graph of the system's potential energy in all three regions. Also draw a dashed line indicating the total energy of the system.

(b) In each of the three regions, are the solutions to Schrödinger's equation real exponentials or complex exponentials? Write down a solution for each of the three regions corresponding to an electron traveling to the right.

Region I:

Region II:

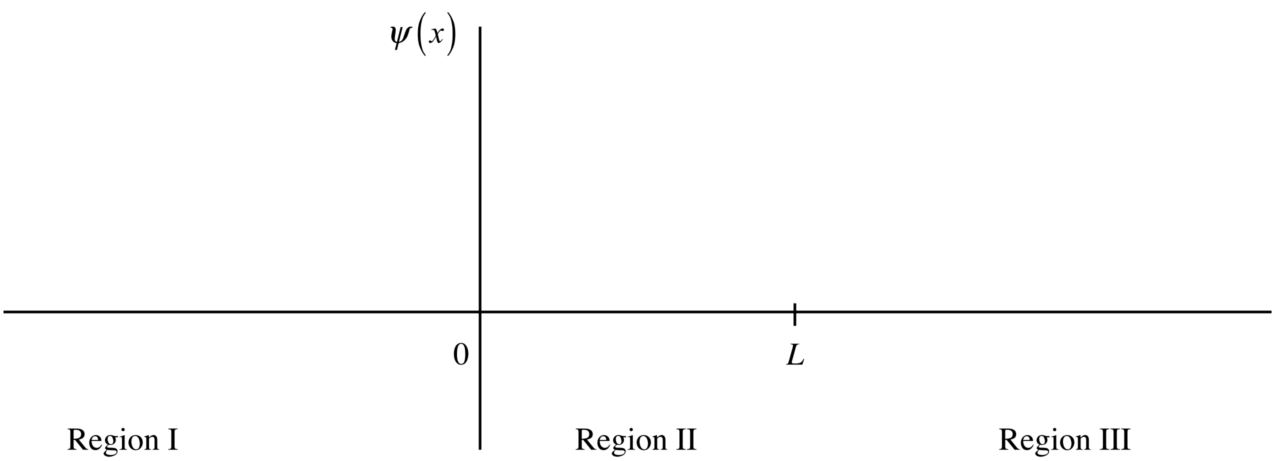
Region III:

(c) How does the kinetic energy of the system (electron) compare in each of the three regions? Rank the kinetic energies in the three regions (KE_1, KE_2 & KE_3) from high to low.

(d) How does the deBroglie wavelength of the electron compare in each of the three regions? Rank the wavelengths in the three regions (λ_1, λ_2, λ_3) from largest to smallest. If the wavelength is not defined in a particular region, then say so.

(e) How does the amplitude of the electron's wave function compare in each of the three regions? [Hint: think about what $|\psi(x)|^2$ tells you in terms of probabilities].

(f) With this information in mind, sketch the ***real part*** of the electron's wave function in all three regions:



(g) Using the solution to part (f), what conclusion can you make about the possible position of the particle? How is this different than a classical particle in the same situation? Can you offer an explanation of why classical objects (people) don't exhibit the same property, called tunneling?

Part F: Quantum particle with *E* > *V* (PhET Simulation)

Download and open the PhET interactive simulation titled "Quantum Tunneling and Wave Packets". In the section titled "Electron Wave Function form:", select "**plane wave**". The top graph shows the system potential energy in purple, and the total energy in green. This potential energy is a good representation of the wire with the air gap.

(a) Observe the plot of the wave function for plane wave solutions for the case where $E > V$. Why is the wave function oscillating up and down?

(b) Now widen the width of the wire gap (where $V > 0$) to 3.5 dashed-lines wide. How does the wavelength of the wave function in this region compare to the wavelength in the region to the left? How about to the region on the right? Lastly, how do the wavelengths in the regions on the left and right compare to each other?

(c) What does your answer to (b) tell you about the kinetic energy of the system (particle) in each of these three regions? Be sure to discuss this with your group members.

(d) Now let's look at the amplitude of the wave function. What does the amplitude of $\psi(x)$ (or $|\psi(x)|^2$) tell you?

(e) If we were to make a measurement of position of the particle, would we be more likely to find it in the region to the left of the air gap or in the air gap (for now ignore the region to the right of the air gap, very much like you did in PART A)?

(f) In this case of $E > V$, explain how measurements of position for a quantum particle compare to taking a photograph of a classical particle.

Part G: Quantum particle with $E < V$ (PhET Simulation)

(a) Now, using the PhET sim, decrease the size of the wire gap to 1 dashed-line wide and increase the height of the potential energy line all the way to the top. What type of function do you see in region 1 and 2?

(b) What type of function is shown inside the wire gap? Hint: It might be more obvious if you look at the wave function when the air gap is very wide… but return to 1 dashed-line wide for the next question!

(c) How do the wavelengths of the wave function on the left and right of the air gap compare to each other? What does that tell you about the kinetic energy of the particle in each of those regions?

(d) Now refer back to PART B with the classical particle. How does the kinetic energy of the classical particle in regions 2 and 3 compare to the kinetic energy of the quantum particle in regions 2 and 3?

(e) What does the amplitude of $\psi(x)$ (or $|\psi(x)|^2$) tell you about finding a particle in regions 2 or 3?

(f) If we were to make a measurement of the particle's position with $E < V$, which region would we be most likely to find it in? Compare this to the case of the classical particle.