The *acceleration* of Q is its radial acceleration $\omega^2 A$ (Fig. 11-6(c)) and its vertical component, equal to the acceleration of P, is

$$a = -\omega^2 A \sin(\omega t + \theta_0).$$

The negative sign must be included because the acceleration is negative whenever the sine of the phase angle is positive, and vice versa. These equations are, of course, just the general equations of harmonic motion. In the special case corresponding to Eqs. (11-17), the inital phase angle is 90° and the reference point Q is at the top of the circle when $t = 0$. If the reference point is at the right-hand end of the horizontal diameter when $t = 0$, then $\theta_0 = 0$ and the motion is described by Eqs. (11-18).

11-5 MOTION OF A BODY SUSPENDED FROM A COIL SPRING

Figure 11-7(a) shows a coil spring of force constant k and no-load length ℓ. When a body of mass m is attached to the spring as in part (b), it hangs in equilibrium with the spring extended by an amount $\Delta \ell$ such that the upward force P exerted by the spring is equal to the weight of the body, mg. But $P = k\,\Delta\ell$, so
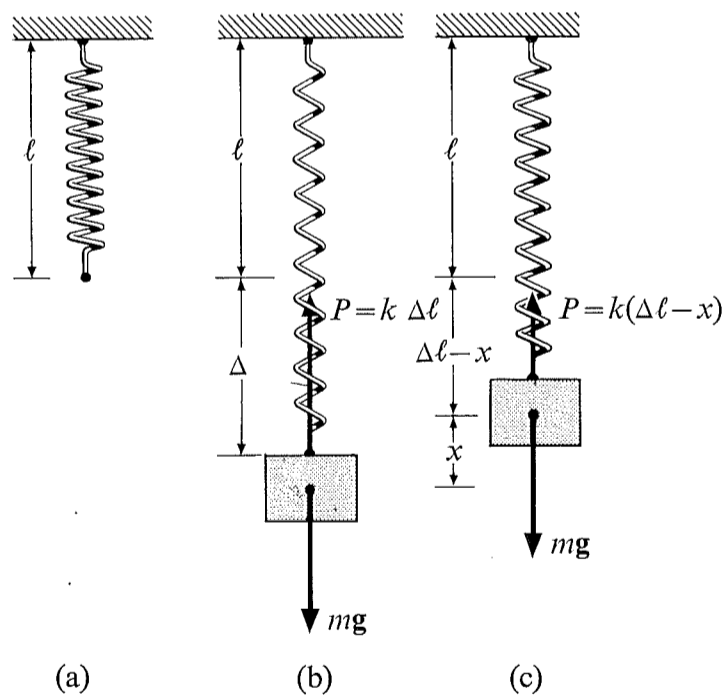
$$k\,\Delta\ell = mg.$$

Fig. 11-7 *The restoring force on a body suspended by a spring is proportional to the coordinate measured from the equilibrium position.*

Now suppose the body is at a distance x *above* its equilibrium position, as in part (c). The extension of the spring is now $\Delta\ell - x$, the upward force it exerts on the body is $k(\Delta\ell - x)$, and the resultant force F on the body is

$$F = k(\Delta\ell - x) - mg = -kx.$$

The resultant force is therefore proportional to the displacement of the body *from its equilibrium position*, and if set in vertical motion the body oscillates with an angular frequency $\omega = \sqrt{k/m}$.

Except in the idealized case of a spring of zero mass, allowance must be made for the fact that the spring also oscillates. However, we cannot simply add the mass of the spring to that of the suspended body because not all portions of the spring oscillate with the same amplitude; the amplitude at the lower end equals that of the suspended body, while that at the upper end is zero. The correction term can be computed as follows.

Let L represent the length of the spring when the body is in its equilibrium position, and let m_s be the mass of the spring. Let us calculate the kinetic energy of the spring at an instant when the velocity of the lower end is v. Consider an element of the spring of length dy, at a distance y below the fixed upper end. The mass of the element, dm_s, is

$$dm_s = \frac{m_s}{L}\,dy.$$

All portions of the spring will be assumed to oscillate in phase and the velocity of the element, v_s, to be proportional to its distance from the fixed end: $v_s = (y/L)v$.

The kinetic energy of the element is

$$dE_k = \frac{1}{2} \cdot dm_s \cdot v_s{}^2 = \frac{1}{2} \cdot \frac{m_s}{L_s}\,dy \cdot \left(\frac{y}{L}v\right)^2,$$

and the total kinetic energy of the spring is

$$E_k = \frac{1}{2} \cdot \frac{m_s v^2}{L_3} \cdot \int_0^L y^2\,dy = \frac{1}{2}\left(\frac{1}{3}m_s\right)v^2.$$

This equals the kinetic energy of a body of mass one-third the mass of the spring, moving with the same velocity as that of the suspended body. In other words, the equivalent mass of the *vibrating system* equals that of the suspended body plus one-third the mass of the spring.