This homework brings together a lot of concepts we've been studying throughout the course: operators, expectation values, probabilities, orthonormality, commutators, Dirac notation, superposition... and probably more that I am failing to list. Unlike last section's homework, which required you to do a lot of integrals, this homework requires none.

1) Commutators with Angular Momentum Operators

(a) Work out the following commutators:

$$\left[\hat{L}_z, \hat{p}_x\right] = i\hbar\hat{p}_y, \qquad \left[\hat{L}_z, \hat{p}_y\right] = -i\hbar\hat{p}_x, \qquad \left[\hat{L}_z, \hat{p}_z\right] = 0$$

Hint: use the definition of L_z from Equation 4.96*, and commutation relations we've used in the past, like Equation* 3.64*.*

(b) Use the results obtained in part a) to evaluate the commutator $\left[\hat{L}_z, \hat{p}^2\right]$.

(c) Show that the z-component of the orbital angular momentum operator expressed in spherical coordinates is

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

Hint: Use chain rule $\frac{\partial}{\partial\phi} = \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z}$ and the definitions of x, y, and z in spherical coordinates, and work backwards from the desired result to show it is equal to L_z in Equation 4.96*.*

(d) Use the fact that $\left[\hat{L}_x, \hat{p}^2\right] = \left[\hat{L}_y, \hat{p}^2\right] = \left[\hat{L}_z, \hat{p}^2\right]$ (you do *not* have to show these relations) and the results obtained in part b) and c) to show that the Hamiltonian $H = (p^2/2m) + V$ commutes with $\hat{L}^2$ and $\hat{L}_z$, provided that $V = V(r)$ depends only on r.

We spent a lot of time working with commutators in this section, so part of this problem is continuing to practice. But more importantly, we were simply told that the eigenfunctions of $\hat{L}^2$ and *$\hat{L}_z$ were spherical harmonics, so from that we could conclude that they must commute with the Hamiltonian. Now we're proving it.*

2) Superposition of Orbital Angular Momentum Eigenstates

Consider a system which is initially in the state

$$\Psi(\theta,\phi,0) = \frac{1}{\sqrt{5}}Y_{1,-1}(\theta,\phi) + \frac{3}{\sqrt{5}}Y_{1,0}(\theta,\phi) + AY_{1,1}(\theta,\phi)$$

where A is a *real* number. ($Y_{l,m}$ are the spherical harmonics, the simultaneous eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$)

(a) Find A such that the state is normalized. Is your answer unique? Explain.

(b) Find $\langle\Psi|\hat{L}_+|\Psi\rangle$.

Hints: We have found that $\hat{L}_\pm Y_{l,m} \propto Y_{l,m\pm1}$. For the problem you need the proportionality factor. It is $\hat{L}_\pm Y_{l,m} = C^\pm_{l,m}Y_{l,m\pm1}$ with $C^\pm_{l,m} = \hbar\sqrt{l(l+1) - m(m\pm1)}$.
Also, use orthonormality!

(c) If $\hat{L}_z$ is measured, what values will one obtain and with what probabilities? What is the expectation value of $\hat{L}_z$?

3) Hydrogen Atom

Complete the worksheet attached with the title "Homework: Angular momentum in quantum mechanics." You are doing Questions 1-4.

- Hint for 3c: The commutator of which two operators gives L_y as a *result*?
- You don't need to do the Optional part on 4c, you can just use the result.

1. Consider an electron in an unknown three-dimensional potential, $V(r, \theta, \varphi)$.

   a. In classical mechanics, under what conditions is the angular momentum a constant of motion (*i.e.*, a conserved quantity)? Explain. (*Hint:* On which variables can the potential depend?)

   b. When can you simultaneously determine l and the total energy of the system?

   When can you simultaneously determine m_l and the energy?

   c. Do your answers to questions a and b, for the classical and quantum mechanical cases, respectively, agree? Explain.

Recall that the eigenfunctions for angular momentum are usually written in terms of the spherical harmonics, $Y_l^m(\theta,\varphi)$. For the remainder of this problem, assume that these functions are ***not*** the energy eigenfunctions of the system.

   d. At $t = 0$, let the wave function for this electron be $\psi(\vec{r}) = f(r)Y_2^{-1}(\theta,\varphi)$. Assume this wave function is normalized, and $f(r)$ is a function of the radial coordinate only.

      i. Determine the possible values of a measurement of the z-component of the orbital angular momentum for this electron at $t = 0$. If you do not have enough information to answer, state so explicitly. Explain.

      ii. Determine the possible values of a measurement of the z-component of the orbital angular momentum for this electron at $t_1 > 0$. If you do not have enough information to answer, state so explicitly. Explain.

      iii. Under what conditions will the probability of measuring $m_l = -1$ be independent of time? Explain.

2. Consider an electron in a hydrogen atom potential.

   a. Assume the state of the particle is given by $|\psi\rangle = |n,l,m_l\rangle = |3,2,-1\rangle$.

      i. Write the wave function for this state in terms of $R_{nl}(r)$ and $Y_l^{m_l}(\theta,\varphi)$ (you do not have to write these out in terms of *r, θ,* and *φ*).

      ii. On which quantum numbers does the energy of this electron depend? If there are any quantum numbers on which the energy *does not* depend, state so explicitly. Explain.

   b. Now assume that the wave function for this particle is given by:

$$\psi(\bar{r}) = \frac{1}{\sqrt{14}}\left(R_{42}(r)Y_2^1(\theta,\varphi) - 3R_{31}(r)Y_1^1(\theta,\varphi) + 2iR_{32}(r)Y_2^1(\theta,\varphi)\right).$$

      i. Write the state of the particle in Dirac notation.

      ii. Assume you were to make a measurement of the energy of this particle. List all possible results of this measurement and the probability associated with each. Explain.

      iii. Assume you were to make a measurement of the orbital angular momentum squared (L^2) for this particle. List all possible results of this measurement and the probability associated with each. Explain.

      iv. Assume you were to make a measurement of the *z*-component of the orbital angular momentum *(L_z)* for this particle. List all possible results of this measurement and the probability associated with each. Explain.

      v. Which, if any, of the three quantities above are well-defined for this particle? Explain.

3. Recall the generalized uncertainty principle, $\sigma_A \sigma_B \geq \frac{1}{2}\left|\left\langle \psi \left| \left[\hat{A},\hat{B}\right] \right| \psi \right\rangle\right|$. Note that the quantity σ_A is known as the *uncertainty* in quantity A (and similarly for B).

   a. Suppose that quantity A is well-defined for a quantum mechanical particle (*i.e.*, there is only one possible outcome of a measurement of that quantity). Would the uncertainty in quantity A, σ_A, be *greater than, less than,* or *equal to* zero? Explain.

   b. Suppose that *all three* components of the angular momentum vector for a particle with $l \neq 0$ are well-defined. Show that this is inconsistent with the uncertainty principle.

   c. Suppose a quantum mechanical particle has a well-defined value of L_z only. Determine the expectation value of L_y, $\left\langle \psi \left| \hat{L}_y \right| \psi \right\rangle$, starting from the left hand side of the uncertainty principle? *(Hint:* What should you choose for quantities A and B?*)*

   Does your answer depend on the value of L^2 for the particle? Explain.

   d. With which of the following statements, if either, do you agree? Explain.

   Student 1: "If one component of angular momentum is well-defined, then the expectation value of the perpendicular components will be zero."

   Student 2: "I agree. That means the probability of measuring zero for those other components will always be the largest, no matter which value we started with."

4. An electron is known to be in the orbital angular momentum state $\left|l,m_l\right\rangle_x = \left|1,1\right\rangle_x$.

   a. List the possible results of a measurement of the z-component of orbital angular momentum. Explain.

   b. *Predict* the ranking of the probabilities of each of the possible results of this measurement from most probable to least probable. Explain.

Note the column vector representations: $\left|1,1\right\rangle_z \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\left|1,0\right\rangle_z \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\left|1,-1\right\rangle_z \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

   c. Optional: Show on your own paper that the operator for L_x, written in the z-basis, is given by: $\frac{\hbar\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. (*Hint:* Recall that $\hat{L}_x = \frac{\hat{L}_+ + \hat{L}_-}{2}$ and take advantage of the following property of the raising and lowering operators: $\hat{L}_\pm\left|l,m_l\right\rangle = \hbar\sqrt{l(l+1)-m_l(m_l \pm 1)}\left|l,m_l \pm 1\right\rangle$.)

   d. Find the column vector in the z-basis that represents the initial orbital angular momentum state $\left|l,m_l\right\rangle_x = \left|1,1\right\rangle_x$. (*Hint:* Use an eigenvalue equation.) Show your work.

   Use your answer to find the probability of each possible value of a measurement in the z-direction. Explain how you arrived at your answer.

e. Repeat your calculation from question d to find the column vector in the z-basis that represents the initial orbital angular momentum state for a particle with the initial state $\left|l, m_l\right\rangle_x = \left|1, 0\right\rangle_x$.

Use your answer to find the probability of each possible value of a measurement in the z-direction on this different particle. Explain how you arrived at your answer.

f. Consider the following statement.

"When we know the angular momentum in the z-direction, we know nothing about the other directions—we have zero information about them. This is a consequence of the uncertainty principle."

Explain why this statement is *incorrect*.