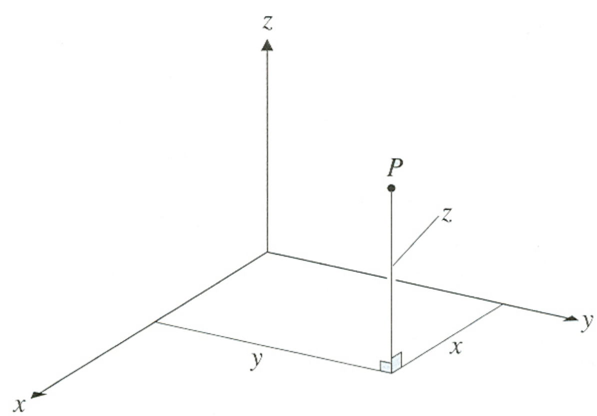
Properties of Coordinate Systems

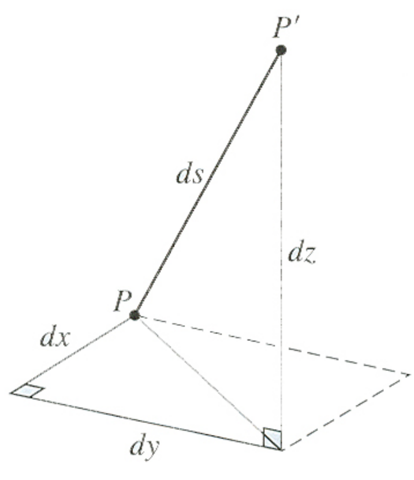
Cartesian Coordinates



Position vector:

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

For Two Neighboring Points P and P':



Displacement between two neighboring points:

$$d\mathbf{s} = d\mathbf{r} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

Distance between two neighboring points (*Found using the Pythagorean Theorem*):

$$ds = |d\mathbf{r}| = \sqrt{dx^2 + dy^2 + dz^2}$$

Primary Curve – the curve obtained when one coordinate variable is allowed to vary while the other two are held fixed.

Primary Length Element – infinitesimal length along the primary curve

Primary Surface – the surface obtained when the coordinate determining the primary length element is held fixed and the other two are allowed to vary.

Primary Element	Primary Curve	Primary Surface	Primary Volume
1st : x	Straight Line (*x-axis*) (y and z fixed, x varies)	*yz*-plane (x fixed , y and z varies)	
2nd : y	Straight Line (*y-axis*) (x and z fixed, y varies)	*xz*-plane (y fixed , x and z varies)	Solid Cube
3rd : z	Straight Line (*z-axis*) (x and y fixed, z varies)	*xy*-plane (z fixed , x and y varies)	

Primary Length Elements	Primary Area Elements	Primary Volume Elements
1st: dx $(\hat{\mathbf{x}})$	$dy\,dz$ $(\hat{\mathbf{x}})$	
2nd: dy $(\hat{\mathbf{y}})$	$dx\,dz$ $(\hat{\mathbf{y}})$	$dx\,dy\,dz$
3rd: dz $(\hat{\mathbf{z}})$	$dx\,dy$ $(\hat{\mathbf{z}})$	

Primary length element vectors are in the direction of their corresponding primary curve.

Primary area element vectors are in the same direction as the primary length element vector (*i.e.* $\perp$ *to their corresponding primary surface*).

Primary volume elements are scalars not vectors and do not have an associated direction.

Conversions to Cartesian Coordinates

Spherical → Cartesian

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Cylindrical → Cartesian

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$
$$z_{Cart} = z_{cyl}$$

Conversions to Cartesian Unit Vectors

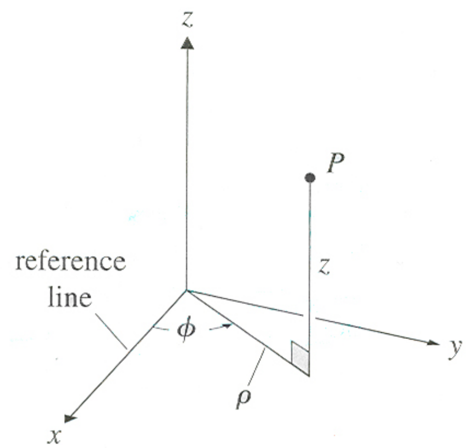
Spherical → Cartesian

$$\hat{\mathbf{r}} = \sin\theta\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$

Cylindrical → Cartesian

$$\hat{\boldsymbol{\rho}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}}_{cyl} = \hat{\mathbf{z}}_{Cart}$$

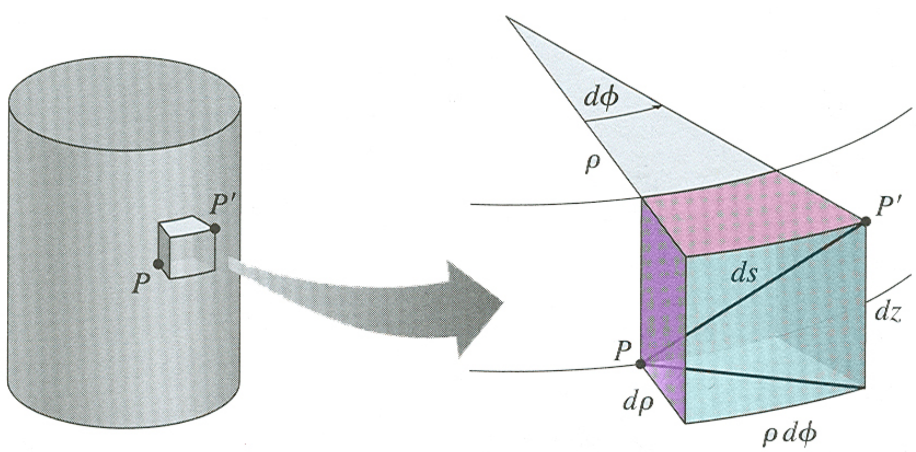
Cylindrical Coordinates



Position vector:

$$\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}$$

For Two Neighboring Points P and P':



Displacement between two neighboring points:

$$d\mathbf{s} = d\mathbf{r} = d\rho\,\hat{\boldsymbol{\rho}} + \rho d\varphi\,\hat{\boldsymbol{\varphi}} + dz\,\hat{\mathbf{z}}$$

Distance between two neighboring points (*Found using the Pythagorean Theorem*):

$$ds = |d\mathbf{r}| = \sqrt{d\rho^2 + \rho^2 d\varphi^2 + dz^2}$$

Primary Curve	Primary Surface	Primary Volume
1st : Rays ⊥ to the *z-axis* (φ and z fixed, ρ varies)	Cylinder centered on the *z-axis* (ρ fixed , φ and z varies)	
2nd : Circle centered on the *z-axis* (ρ and z fixed, φ varies)	Half-plane from *z-axis* (φ fixed , ρ and z varies)	Solid Cylinder
3rd : Straight line (*z-axis*) (ρ and φ fixed, z varies)	Plane ⊥ to the *z-axis* (z fixed , ρ and φ varies)	

Primary Length Elements	Primary Area Elements	Primary Volume Elements
1st: $d\rho$ $(\hat{\boldsymbol{\rho}})$	$\rho\, d\varphi\, dz$ $(\hat{\boldsymbol{\rho}})$ (teal surface)	
2nd: $\rho\, d\varphi$ $(\hat{\boldsymbol{\varphi}})$	$d\rho\, dz$ $(\hat{\boldsymbol{\varphi}})$ (purple surface)	$\rho\, d\rho\, d\varphi\, dz$
3rd: dz $(\hat{\mathbf{z}})$	$\rho\, d\rho\, d\varphi$ $(\hat{\mathbf{z}})$ (pink surface)	

Conversions to Cylindrical Coordinates

Cartesian → Cylindrical

$$\rho = \sqrt{x^2 + y^2}$$

$$\tan\varphi = \frac{y}{x} \quad \Rightarrow \quad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z_{cyl} = z_{Cart}$$

Spherical → Cylindrical

$$\rho = r\sin\theta$$

$$\varphi_{cyl} = \varphi_{sph}$$

$$z = r\cos\theta$$

Conversions to Cylindrical Unit Vectors

Cartesian → Cylindrical

$$\hat{\mathbf{x}} = \cos\varphi\,\hat{\boldsymbol{\rho}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = \sin\varphi\,\hat{\boldsymbol{\rho}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

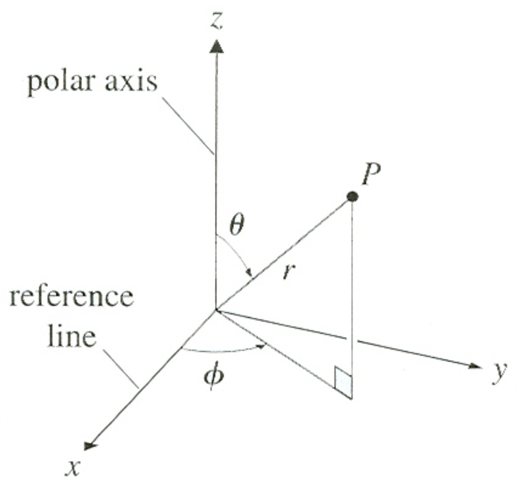
Spherical → Cylindrical

$$\hat{\mathbf{r}} = \sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\rho}} - \sin\theta\,\hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\varphi}}_{sph} = \hat{\boldsymbol{\varphi}}_{cyl}$$

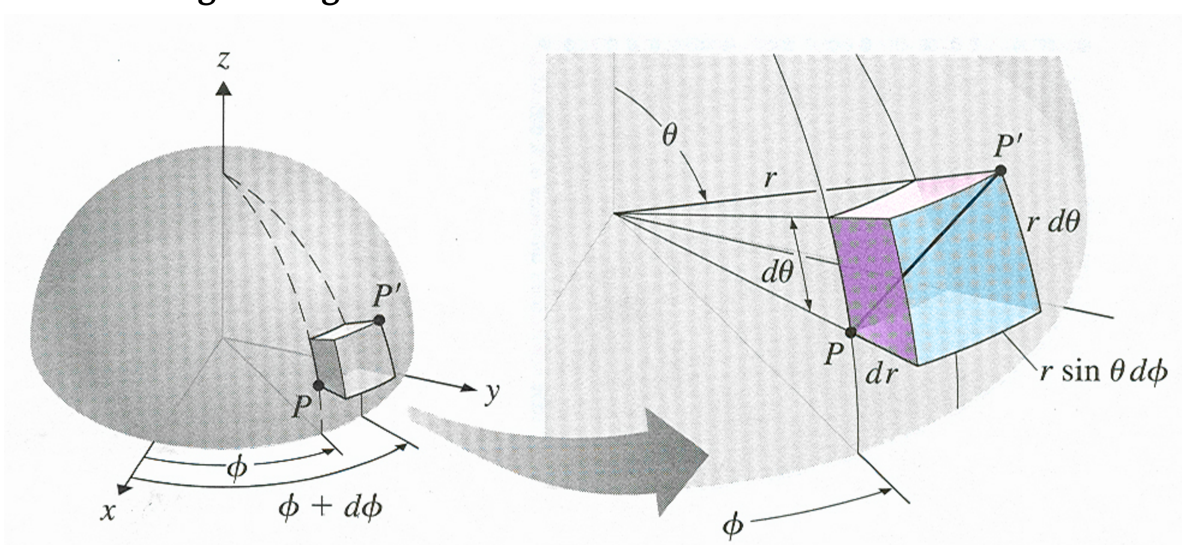
Spherical Coordinates



Position vector:

$$\mathbf{r} = r\,\hat{\mathbf{r}}$$

For Two Neighboring Points P and P':



Displacement between two neighboring points:

$$d\mathbf{s} = d\mathbf{r} = dr\,\hat{\mathbf{r}} + r d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta d\varphi\,\hat{\boldsymbol{\varphi}}$$

Distance between two neighboring points (*Found using the Pythagorean Theorem*):

$$ds = |d\mathbf{r}| = \sqrt{dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2}$$

Primary Element	Primary Curve	Primary Surface	Primary Volume
1st : r	Rays from the origin (θ and φ fixed, r varies)	Sphere (r fixed , θ and φ varies)	
2nd : θ	Half circle (r and φ fixed, θ varies)	Cone of half angle θ (θ fixed , r and φ varies)	Solid Sphere
3rd : φ	Circle centered on polar axis (r and θ fixed, φ varies)	Half-plane from *z*-axis (φ fixed , r and θ varies)	

Primary Length Elements		Primary Area Elements			Primary Volume Elements
1st: dr	($\hat{\mathbf{r}}$)	$r^2 \sin\theta \, d\theta \, d\varphi$	($\hat{\mathbf{r}}$)	(teal)	
2nd: $r \, d\theta$	($\hat{\boldsymbol{\theta}}$)	$r \sin\theta \, dr \, d\varphi$	($\hat{\boldsymbol{\theta}}$)	(pink)	$r^2 \sin\theta \, dr \, d\theta \, d\varphi$
3rd: $r \sin\theta \, d\varphi$	($\hat{\boldsymbol{\varphi}}$)	$r \, dr \, d\theta$	($\hat{\boldsymbol{\varphi}}$)	(purple)	

Note: The $r \sin\theta$ term is the distance from the polar axis to the projection of point P into the *xy*-plane.

Conversions to Spherical Coordinates

Cartesian → Spherical

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z} \quad \text{or} \quad \cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\tan\varphi = \frac{y}{x}$$

➔

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Cylindrical → Spherical

$$r = \sqrt{\rho^2 + z^2}$$

$$\tan\theta = \frac{\rho}{z} \quad \text{or} \quad \cos\theta = \frac{z}{\sqrt{\rho^2 + z^2}}$$

$$\varphi_{sph} = \varphi_{cyl}$$

➔

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{\rho^2 + z^2}}\right)$$

Conversions to Spherical Unit Vectors

Cartesian → Spherical

$$\hat{\mathbf{x}} = \sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = \sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\theta\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

Cylindrical → Spherical

$$\hat{\boldsymbol{\rho}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}$$

$$\hat{\boldsymbol{\varphi}}_{cyl} = \hat{\boldsymbol{\varphi}}_{sph}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$