APPENDIX G

The Greek Alphabet

A, α	Alpha	N, ν	Nu
B, β	Beta	Ξ, ξ	Xi
Γ, γ	Gamma	O, o	Omicron
Δ, δ	Delta	Π, π	Pi
E, ϵ	Epsilon	P, ρ	Rho
Z, ζ	Zeta	Σ, σ	Sigma
H, η	Eta	T, τ	Tau
Θ, θ	Theta	Υ, υ	Upsilon
I, ι	Iota	Φ, ϕ	Phi
K, κ	Kappa	X, χ	Chi
Λ, λ	Lambda	Ψ, ψ	Psi
M, μ	Mu	Ω, ω	Omega

APPENDIX H

Handy Trigonometric Identities

$$1\,\text{rad} = \frac{180}{\pi} = 57.296^\circ$$

$$1^\circ = \frac{\pi}{180} = 0.17453\,\text{rad}$$

Values of sine and cosine for multiples of π

$$\sin n\pi = 0 \qquad n = 0,\ 1,\ 2,\ 3,\ 4,\ \ldots$$

$$\cos n\pi = (-1)^n$$

$$\sin \tfrac{n\pi}{2} = \tfrac{1}{2}(i)^{n+1}[(-1)^n - 1]$$

$$\cos \tfrac{n\pi}{2} = \tfrac{1}{2}(i)^n[1 + (-1)^n]$$

$$\sin \tfrac{n\pi}{4} = 0 \qquad n = 0,\ 4,\ 8,\ 12,\ 16,\ \ldots \qquad 4m\ (m = \text{integer})$$

$$\sin \tfrac{n\pi}{4} = (-1)^{\frac{1}{4}(n-2)} \qquad n = 2,\ 6,\ 10,\ 14,\ 18,\ \ldots \qquad 4m + 2\ (m = \text{integer})$$

$$\sin \tfrac{n\pi}{4} = \tfrac{1}{\sqrt{2}}(-1)^{\frac{1}{8}(n^2+4n+11)} \qquad n = 1,\ 3,\ 5,\ 7,\ \ldots$$

Sums and differences of angles

α =	$90^\circ \pm \beta$	$180^\circ \pm \beta$	$270^\circ \pm \beta$	$360^\circ \pm \beta$
$\sin\alpha$ =	$+\cos\beta$	$\mp\sin\beta$	$-\cos\beta$	$\pm\sin\beta$
$\cos\alpha$ =	$\mp\sin\beta$	$-\cos\beta$	$\pm\sin\beta$	$+\cos\beta$
$\tan\alpha$ =	$\mp\cot\beta$	$\pm\tan\beta$	$\mp\cot\beta$	$\pm\tan\beta$
$\cot\alpha$ =	$\mp\tan\beta$	$\pm\cot\beta$	$\mp\tan\beta$	$\mp\cot\beta$

Exponential definition of trigonometric functions

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

Fundamental Trigonometric Relations Involving Only One Angle

$$\sin^2\alpha + \cos^2\alpha = 1 \qquad \sec^2\alpha - \tan^2\alpha = 1 \qquad \operatorname{cosec}^2\alpha - \cot^2\alpha = 1$$

$$\sin\alpha\cdot\operatorname{cosec}\alpha = 1, \qquad \cos\alpha\cdot\sec\alpha = 1, \qquad \tan\alpha\cdot\cot\alpha = 1$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+\cot^2\alpha}}$$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \frac{\cot\alpha}{\sqrt{1+\cot^2\alpha}} = \frac{1}{\sqrt{1+\tan^2\alpha}}$$

$$\tan\alpha = \sqrt{\sec^2\alpha - 1} = \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}} = \frac{1}{\sqrt{\operatorname{cosec}^2\alpha - 1}}$$

$$\cot\alpha = \sqrt{\operatorname{cosec}^2\alpha - 1} = \frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}} = \frac{1}{\sqrt{\sec^2\alpha - 1}}$$

$$\tan\left(\alpha + \tfrac{\pi}{4}\right) = \frac{1+\tan\alpha}{1-\tan\alpha}$$

$$\cot\left(\tfrac{\pi}{4} - \beta\right) = \frac{\cot\beta + 1}{\cot\beta - 1}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\cot\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$$

$$\sin^2\alpha = \tfrac{1}{2}(1-\cos 2\alpha),$$
$$\cos^2\alpha = \tfrac{1}{2}(1+\cos 2\alpha)$$

$$\sin^3\alpha = \tfrac{1}{4}(3\sin\alpha - \sin 3\alpha)$$
$$\cos^3\alpha = \tfrac{1}{4}(\cos 3\alpha + 3\cos\alpha)$$

Fundamental Trigonometric Relations Involving Two or More Angles

$$\sin\alpha \pm \sin\beta = 2\sin\tfrac{1}{2}(\alpha\pm\beta)\cos\tfrac{1}{2}(\alpha\mp\beta)$$

$$\cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$$

$$\cos\alpha - \cos\beta = -2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta)$$

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$$

$$\cot\alpha \pm \cot\beta = \frac{\pm\sin(\alpha\pm\beta)}{\sin\alpha\sin\beta}$$

$$\sin^2\alpha - \sin^2\beta = \sin(\alpha+\beta)\cdot\sin(\alpha-\beta)$$

$$\cos^2\alpha - \cos^2\beta = -\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)$$

$$\cos^2\alpha - \sin^2\beta = \cos(\alpha+\beta)\cdot\cos(\alpha-\beta)$$

$$\frac{\sin\alpha \pm \sin\beta}{\cos\alpha + \cos\beta} = \tan\tfrac{1}{2}(\alpha\pm\beta)$$

$$\frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}$$

$$\frac{\sin\alpha \pm \sin\beta}{\cos\alpha - \cos\beta} = -\cot\tfrac{1}{2}(\alpha \pm \beta)$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$$

$$\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$$

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\cos\gamma - \sin\alpha\cos\beta\sin\gamma - \cos\alpha\sin\beta\sin\gamma$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$$

$$\sin\alpha\sin\beta\sin\gamma = \tfrac{1}{4}[\sin(\alpha+\beta-\gamma) + \sin(\beta+\gamma-\alpha) + \sin(\gamma+\alpha-\beta) - \sin(\alpha+\beta+\gamma)]$$

$$\sin\alpha\cos\beta\cos\gamma = \tfrac{1}{4}[\sin(\alpha+\beta-\gamma) - \sin(\beta+\gamma-\alpha) + \sin(\gamma+\alpha-\beta) + \sin(\alpha+\beta+\gamma)]$$

$$\sin\alpha\sin\beta\cos\gamma = \tfrac{1}{4}[-\cos(\alpha+\beta-\gamma) + \cos(\beta+\gamma-\alpha) + \cos(\gamma+\alpha-\beta) - \cos(\alpha+\beta+\gamma)]$$

$$\cos\alpha\cos\beta\cos\gamma = \tfrac{1}{4}[\cos(\alpha+\beta-\gamma) + \cos(\beta+\gamma-\alpha) + \cos(\gamma+\alpha-\beta) + \cos(\alpha+\beta+\gamma)]$$

Hyperbolic Functions

$$\sinh u = \frac{e^u - e^{-u}}{2} \qquad \cosh u = \frac{e^u + e^{-u}}{2}$$

$$\tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}} \qquad \coth u = \frac{e^u + e^{-u}}{e^u - e^{-u}}$$

$$\operatorname{csch} u = \frac{1}{\sinh u} \qquad \operatorname{sech} u = \frac{1}{\cosh u}$$

$$\coth u = \frac{1}{\tanh u}$$

$\sinh(-u) = -\sinh u$ odd function

$\cosh(-u) = \cosh u$ even function

$\tanh(-u) = -\tanh u$ odd function

$\tanh(-u) = -\coth u$ odd function

Fundamental Relations Involving One Angle

$$\sinh u = \frac{\tanh u}{\operatorname{sech} u} \qquad \cosh u = \frac{\coth u}{\operatorname{cosech} u} \qquad \tanh u = \frac{\sinh u}{\cosh u}$$

$$\operatorname{cosech} u = \frac{\operatorname{sech} u}{\tanh u} \qquad \operatorname{sech} u = \frac{\operatorname{cosech} u}{\coth} \qquad \coth u = \frac{\cosh u}{\sinh u}$$

$$\sinh u = \tanh u\cosh u \qquad \cosh u = \coth u\sinh u$$

$$\tanh u = \sinh u\operatorname{sech} u \qquad \coth u = \cosh u\operatorname{cosech} u$$

$$\operatorname{sech} u = \operatorname{cosech} u\tanh u \qquad \operatorname{cosech} u = \operatorname{sech} u\coth u$$

$$\cosh^2 u - \sinh^2 u = 1 \qquad \tanh^2 u + \operatorname{sech}^2 u = 1$$

$$\coth^2 u - \operatorname{csch}^2 u = 1$$

$$\operatorname{csch}^2 u - \operatorname{sech}^2 u = \operatorname{csch}^2 u\operatorname{sech}^2 u$$

Fundamental Relations Involving Two or More Angles

$$\sinh(u+v) = \sinh u\cosh v + \cosh u\sinh v$$

$$\sinh(u-v) = \sinh u\cosh v - \cosh u\sinh v$$

$$\cosh(u+v) = \cosh u\cosh v + \sinh u\sinh v$$

$$\cosh(u-v) = \cosh u\cosh v - \sinh u\sinh v$$

Multiples of One Angle

$$\sinh 2u = 2\sinh u\cosh u = \frac{2\tanh u}{1-\tanh^2 u}$$

$$\cosh 2u = \cosh^2 u + \sinh^2 u = 2\cosh^2 u - 1 = 1 + 2\sinh^2 u = \frac{1+\tanh^2 u}{1-\tanh^2 u}$$

$$\tanh 2u = \frac{2\tanh u}{1+\tanh^2 u} \qquad \coth 2u = \frac{\coth^2 u + 1}{2\coth u}$$

$$\sinh u + \sinh v = 2 \sinh \frac{1}{2}(u + v) \cosh \frac{1}{2}(u - v)$$

$$\sinh u - \sinh v = 2 \cosh \frac{1}{2}(u + v) \sinh \frac{1}{2}(u - v)$$

$$\cosh u + \cosh v = 2 \cosh \frac{1}{2}(u + v) \cosh \frac{1}{2}(u - v)$$

$$\cosh u - \cosh v = 2 \sinh \frac{1}{2}(u + v) \sinh \frac{1}{2}(u - v)$$

$$\sinh u \cosh v = \frac{1}{2} \sinh(u + v) + \frac{1}{2} \sinh(u - v)$$

$$\cosh u \sinh v = \frac{1}{2} \sinh(u + v) - \frac{1}{2} \sinh(u - v)$$

$$\cosh u \cosh v = \frac{1}{2} \cosh(u + v) + \frac{1}{2} \cosh(u - v)$$

$$\sinh u \sinh v = \frac{1}{2} \cosh(u + v) - \frac{1}{2} \cosh(u - v)$$

$$\sinh^2 u = \frac{1}{2}(\cosh 2u - 1)$$

$$\cosh^2 u = \frac{1}{2}(\cosh 2u + 1)$$

$$(\cosh u + \sinh u)^n = \cosh nu + \sinh nu$$

Relations Between Hyperbolic and Trigonometric Functions

$$\sinh iu = i \sin u, \qquad \sinh u = -i \sin iu$$

$$\cosh iu = \cos u, \qquad \cosh u = \cos iu$$

$$\tanh iu = i \tan u, \qquad \tanh u = -i \tan iu$$

Every hyperbolic relation may be obtained from the corresponding trigonometric relation by replacing $\sin \alpha$ by $i \sinh u$ and $\cos \alpha$ by $\cosh u$.

APPENDIX I

Integrals You Should Know

Indefinite Integrals

$$\int x \sin bx \, dx = \frac{1}{b^2} \sin bx - \frac{x}{b} \cos bx$$

$$\int x e^{bx} \, dx = \frac{e^{bx}}{b^2}(bx - 1)$$

$$\int x^2 e^{bx} \, dx = e^{bx} \left(\frac{x^2}{b} - \frac{2x}{b^2} + \frac{2}{b^3} \right)$$

$$\int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx$$

$$\int \sin x \, dx = -\cos x$$

$$\int \sin \frac{x}{a} dx = -a \cos \frac{x}{a}$$

$$\int x \sin x \, dx = \sin x - x \cos x$$

$$\int x^2 \sin x \, dx = 2x \sin x - (x^2 - 2) \cos x$$

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{x}{2} - \frac{\sin x \cos x}{2}$$

$$\int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8}$$

$$\int x^2 \sin^2 x \, dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right) \sin 2x - \frac{x}{4} \cos 2x$$

$$\int \sin^3 x\, dx = \frac{\cos^3 x}{3} - \cos x$$

$$\int x \cos x\, dx = \cos x + x \sin x$$

$$\int x^2 \cos x\, dx = 2x \cos x + (x^2 - 2) \sin x$$

$$\int x^2 \cos^2 x\, dx = \frac{x^3}{6} + \left(\frac{x^2}{4} - \frac{1}{8}\right) \sin 2x + \frac{x}{4} \cos 2x$$

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{x}{2} + \frac{\sin x \cos x}{2}$$

$$\int x \cos^2 x\, dx = \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}$$

$$\int x^2 \cos^2 x\, dx = \frac{x^3}{6} + \left(\frac{x^2}{4} - \frac{1}{8}\right) \sin 2x + \frac{x \cos 2x}{4}$$

$$\int \cos^3 x\, dx = \sin x - \frac{\sin^3 x}{3}$$

$$\int x \cos^3 x\, dx = \frac{x \sin 3x}{12} + \frac{\cos 3x}{36} + \frac{3}{4} x \sin x + \frac{3}{4} \cos x$$

$$\int \sin x \cos^2 x\, dx = -\frac{\cos^3 x}{3}$$

$$\int \sin^2 x \cos x\, dx = \frac{\sin^3 x}{3}$$

$$\int \tan^2 x\, dx = \tan x - x$$

$$\int \cot^2 x\, dx = -\cot x - x$$

$$\int x^2 e^{ax} dx = e^{ax} \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$

Integrals from 0 *to* ∞ *[for* $a > 0$*]*

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}} = \frac{\Gamma(n+1)}{a^{n+1}}, \quad n > -1$$

$$\int_0^\infty e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty x\, e^{-ax^2}\, dx = \frac{1}{2a}$$

$$\int_0^\infty x^2\, e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_0^\infty x^3\, e^{-ax^2}\, dx = \frac{1}{2a^2}$$

$$\int_0^\infty x^{2n}\, e^{-ax^2}\, dx = \frac{1 \cdot 3 \cdots (2n-1)}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}} \quad [n = 1, 2, 3, \ldots]$$

$$\int_0^\infty x^{2n+1}\, e^{-ax^2}\, dx = \frac{n!}{2a^{n+1}} \quad [n = 0, 1, 2, \ldots]$$

$$\int_0^\infty \frac{dx}{(a^2 + x^2)^n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} \frac{\pi}{2a^{2n-1}}, \quad [a > 0; n = 2, 3, \ldots]$$

$$\int_0^\infty e^{-ax} \cos mx\, dx = \frac{a}{a^2 + m^2}$$

Integrals from 1 *to* ∞ *[for* $a > 0$*]*

$$\int_1^\infty e^{-ax}\, dx = \frac{e^{-a}}{a}$$

$$\int_1^\infty x\, e^{-ax}\, dx = \frac{e^{-a}}{a^2}(1 + a)$$

$$\int_1^\infty x^2\, e^{-ax}\, dx = \frac{2\, e^{-a}}{a^3}\left(1 + a + \frac{a^2}{2}\right)$$

Integrals from -1 *to* $+1$

$$\int_{-1}^{+1} e^{-ax}\, dx = \frac{1}{a}\left(e^a - e^{-a}\right)$$

$$\int_{-1}^{+1} x\, e^{-ax}\, dx = \frac{1}{a^2}\left[e^a - e^{-a} - a\left(e^a + e^{-a}\right)\right]$$

$$\int_{-1}^{+1} x^n\, e^{-ax}\, dx = (-1)^{n+1} A_n(-a) - A_n(a)$$

where

$$A_n(a) \equiv \int_0^\infty x^n\, e^{-ax}\, dx = \frac{n!\, e^{-a}}{a^{n+1}} \sum_{k=0}^{n} \frac{a^k}{k!}$$

$$\int_{-1}^{+1} x^n\, dx = \begin{cases} 0 & n = 1, 3, 5, \ldots \\ \dfrac{2}{n+1} & n = 0, 2, 4, \ldots \end{cases}$$

Integrals from 0 to 1 [for $a > 0$]

$$\int_0^1 x\, e^{-ax}\, dx = \frac{1}{a^2}\, [1 - e^{-a}(1+a)]$$

$$\int_0^1 e^{-ax}\, dx = \frac{1}{a}\, (1 - e^{-a})$$

$$\int_0^1 x^2\, e^{-ax}\, dx = \frac{2}{a^3} \left[1 - e^{-a} \left(1 + a + \frac{a^2}{2}\right)\right]$$

Integrals from 0 to $\pi/2$

$$\int_0^{\pi/2} \sin^2 mx\, dx = \int_0^{\pi/2} \cos^2 mx\, dx = \frac{\pi}{4} \quad [m = 1, 2, \ldots]$$

Integrals from 0 to π [m and n non-zero integers]

$$\int_0^\pi \sin^2 x\, dx = \int_0^\pi \cos^2 x\, dx = \frac{\pi}{2}$$

$$\int_0^\pi \sin mx \sin nx\, dx = \begin{cases} 0 & [m \neq n] \\ \dfrac{\pi}{2} & [m = n] \end{cases}$$

$$\int_0^\pi \cos mx \cos nx\, dx = \begin{cases} 0 & [m \neq n] \\ \dfrac{\pi}{2} & [m = n] \end{cases}$$

$$\int_0^\pi \sin mx \cos nx\, dx = \begin{cases} 0 & [m = n] \\ 0 & [m \neq n; (m+n) \text{ even}] \\ \dfrac{2m}{m^2 - n^2} & [m \neq n; (m+n) \text{ odd}] \end{cases}$$

Still More Integrals [for $a > 0$]

$$\int_y^\infty x^n\, e^{-ax}\, dx = \frac{n!\, e^{-ay}}{a^{n+1}} \sum_{k=0}^{n} \frac{(ay)^k}{k!} \quad [n = 0, 1, 2, \ldots]$$

$$\int_t^\infty z^n\, e^{-az}\, dz = \frac{n!}{a^{n+1}}\, e^{-at} \left(1 + at + \frac{a^2 t^2}{2!} + \cdots + \frac{a^n t^n}{n!}\right), \quad [n = 0, 1, 2, \ldots]$$

$$\int_0^{2\pi} \sin^2 mx\, dx = \int_0^{2\pi} \cos^2 mx\, dx = \pi, \quad [m = 1, 2, \ldots]$$

APPENDIX J

Integrals Involving the Gaussian Function

General Results

$$\boxed{I_n \equiv \int_{-\infty}^{\infty} x^n\, e^{-(a+ib)x^2}\, dx}$$

$$\left.\begin{aligned} I_{2m+1} &= 0 \\ I_{2m} &= (-1)^m \left(\frac{d}{da}\right)^m \sqrt{\frac{\pi}{a+ib}} \end{aligned}\right\} \qquad m = 0, 1, 2, 3, \ldots$$

$$\boxed{J_n \equiv \int_{-\infty}^{\infty} x^n\, e^{-\alpha x^2 - \beta x}\, dx}$$

$$J_{2m+1} = (-1)^{2m+1} \left(\frac{\partial}{\partial \beta}\right)^{2m+1} \left(\sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/(4\alpha)}\right)$$

$$J_{2m} = (-1)^m \left(\frac{\partial}{\partial \alpha}\right)^m \left(\sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/(4\alpha)}\right)$$

$$J_{m+1} = -\frac{\partial}{\partial \beta} J_m$$

Specific Results

$$I_0 = \int_{-\infty}^{\infty} e^{-(a+ib)x^2}\, dx = \sqrt{\frac{\pi}{a+ib}}$$

$$I_2 = \int_{-\infty}^{\infty} x^2\, e^{-(a+ib)x^2}\, dx = \frac{\sqrt{\pi}}{2}\, (a+ib)^{-3/2}$$

$$J_0 = \int_{-\infty}^{\infty} e^{-\alpha x^2 - \beta x}\, dx = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/(4\alpha)}$$

$$J_1 = \int_{-\infty}^{\infty} x\, e^{-\alpha x^2 - \beta x}\, dx = -\frac{\beta}{2\alpha} \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/(4\alpha)}$$

$$J_2 = \int_{-\infty}^{\infty} x^2\, e^{-\alpha x^2 - \beta x}\, dx = \left(\frac{1}{2\alpha} + \frac{\beta^2}{4\alpha^2}\right) \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/(4\alpha)}$$

APPENDIX K

Review of Complex Numbers

$$i = \sqrt{-1}$$

Real and Imaginary Parts

If $z = x + iy$, then $x = \Re(z)$ and $y = \Im(z)$.

Polar Representation of a Complex Number

If we plot $x = \Re(z)$ and $y = \Im(z)$, then we can represent z by the polar coordinates r and θ, where

$$r = |z| = \text{distance of the point } z \text{ from the origin ("modulus" of } z)$$

$$\theta = \tan^{-1}\frac{y}{x} = \text{angle that the vector to } z \text{ makes with the } x \text{ axis}$$

The following relationships hold:

$$x = r\cos\theta \qquad |z| = r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

All About the Complex Conjugate

$$\text{If } z = x + iy, \text{ then } z^* = x - iy$$

If $\hat{A}$ is an operator, then $(\hat{A}\Psi)^* = \hat{A}^\dagger\Psi^*$
A real number is one that is equal to its complex conjugate, *i.e.*,

$$z \text{ real} \quad \Longrightarrow \quad z = z^*$$

de Moivre's Theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Simple Algebra Involving Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$
$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

Multiples of i

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = +1, \quad i^5 = ii^4 = i, \quad i^6 = ii^5 = i^2 = -1$$
$$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+k} = i^k$$
$$1/i = -i, \quad 1/i^2 = i^2 = -1, \quad 1/i^3 = 1/ii^2 = -1/i = i$$

APPENDIX L

The Dirac Delta Function

Basic Properties

Definition	$\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0)$
Nature	$\delta(x-x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases}$
Normalization	$\int_{-\infty}^{\infty} \delta(x-x_0)\, dx = 1$
$\delta(x)$ is real	$\delta^*(x) = \delta(x)$
$\delta(x)$ is even	$\delta(x) = \delta(-x)$

Additional Properties

$$\delta(ax) = \frac{1}{a}\delta(x) \qquad \text{for } a > 0$$

$$\int \delta'(x) f(x)\, dx = -f'(0), \qquad \text{where } \delta'(x) = \frac{d}{dx}\delta(x)$$

$$\delta'(-x) = -\delta'(x)$$

$$x\delta(x) = 0$$

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)] \qquad \text{for } a > 0$$

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

$$\int \delta(x-b)\delta(a-x)\, dx = \delta(a-b)$$

$$\int_{-\infty}^{\infty} \frac{d^m \delta(x)}{dx^m} f(x)\, dx = (-1)^m \left[\frac{d^m f}{dx^m}\right]_{x=0}$$

$$\delta[f(x)] = \frac{1}{|df/dx|_{x=x_0}} \delta(x-x_0), \qquad x_0 = \text{a root of } f(x)$$

Representations

$$\delta(x) = \lim_{\alpha \to \infty} \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2} \qquad \text{[normalized Gaussian of width } (2\alpha)^{-1/2}\text{]}$$

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx}\, dk \qquad \text{[Fourier transform of } 1/\sqrt{2\pi}\text{]}$$

$$\delta(x) = \lim_{\alpha \to 0} \frac{1}{\pi} \frac{\sin(x/\alpha)}{x} \qquad \text{[diffraction amplitude of width } \propto 1/\alpha\text{]}$$

$$\delta(x) = \frac{d}{dx}\theta(x)$$

where $\theta(x)$ is the step function, defined by

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$\delta(x-x') = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi}\, 2^n\, n!} \exp\left[-\left(\frac{x^2 + x'^2}{2}\right)\right] H_n(x) H_n(x')$$

where $H_n(x)$ is the n^{th}-order Hermite Polynomial.

$$\delta(x-x') = \frac{1}{\pi} \int_0^{\infty} \cos k(x-x')\, dk$$

$$\delta(x-x') = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \exp[in(x-x')]$$

$$\delta(x-x') = \frac{1}{2\pi}\left[1 + \sum_{1}^{\infty} 2\cos n(x-x')\right]$$

$$\delta(x - x') = \lim_{\epsilon \to 0} \frac{e^{-(x-x')^2/\epsilon^2}}{\epsilon\sqrt{\pi}}$$

$$\delta(x - x') = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{(x - x')^2 + \epsilon^2}$$

$$\delta(x) = \frac{1}{2L} + \frac{1}{L} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)$$ Fourier series for region $+L$ to $-L$