ALGEBRA REVIEW MODULE

Introduction

One widely used algebra textbook* begins with the sentence: "Algebra is really arithmetic in disguise." Arithmetic makes use of specific numbers; algebra develops general results that can be applied regardless of what the particular numbers are. That makes algebra much more powerful than arithmetic. But if you want to *apply* a given algebraic result, you have to replace the *x*s and *y*s by specific numbers, and you have to know how to handle numerical quantities. So this algebra review begins with some stuff about numbers as such.

I. SCIENTIFIC NOTATION

Writing numbers in what is called "scientific notation" is an absolute necessity in science and engineering. You will be using it all the time. It is based on the fact that any positive number, however large or however small, can be written as a number between 1 and 10 multiplied by a power of 10. (And, for negative numbers, we simply put a - sign in front.)

Begin by recalling that:

$$10^1 = 10$$
$$10^2 = 10 \cdot 10 = 100$$
$$10^3 = 10 \cdot 10 \cdot 10 = 1000$$
$$\text{etc.}$$

To *multiply* together any two powers of 10, we simply *add* the exponents:

$$(10^m)(10^n) = 10^{(m+n)}.$$

[These and related matters are discussed in more detail in the module on Exponentials & Logarithms.]

To *divide* one power of 10 by another, we *subtract* the exponents:

$$\frac{10^m}{10^n} = 10^{(m-n)}.$$

Thus $10^9/10^4 = 10^{(9-4)} = 10^5 = 100{,}000$.

It is implicit in this that the reciprocal of a positive power of 10 is an equal negative power:

$$\frac{1}{10^n} = 10^{-n}.$$

The powers-of-10 notation also defines what is meant by 10^0:

$$10^0 = \frac{10^n}{10^n} = 1.$$

This is all very simple, but you need to be careful when negative powers of 10 are involved. Give yourself some practice by doing the following exercises (next page):

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*Hughes-Hallett, *The Math Workshop: Algebra*, W. W. Norton Co., New York, 1980.

Exercise I.1

Write the following numbers in scientific notation:
(a) 80,516; (b) 0.0751; (c) 3,520,000; (d) 0.000 000 081.

NOTE: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

[**A Note about Notation**: We are using the center dot, e.g., $1.5 \cdot 10^4$, when we write a number as the product of a decimal with a power of 10. Perhaps you are accustomed to using the multiplication sign, x, for this purpose, and you will find that both conventions are widely used. We prefer to use the center dot (especially in algebra and, later, in calculus) or, in some places, a parenthesis, to avoid any possible confusion with the variable *x*.]

Exercise I.2

Evaluate: (a) $10^9 \cdot 10^{-3}$; (b) $10^7/10^{-4}$; (c) $10^{-19}/10^{-34}$.

Answers?: You don't need *us* to provide them! Just get the practice.

Now consider multiplying or dividing two numbers that are *not* pure powers of 10.

Take, for example, the numbers $2.6 \cdot 10^3$ and $5.3 \cdot 10^4$.

Their product is given by:
$(2.6 \cdot 10^3)(5.3 \cdot 10^4) = (2.6)(5.3)(10^3 \cdot 10^4) = 13.8 \cdot 10^{(3+4)} = 13.8 \cdot 10^7 = 1.38 \cdot 10^8$.

Their quotient is given by:

$$\frac{2.6 \cdot 10^3}{5.3 \cdot 10^4} = \left(\frac{2.6}{5.3}\right) 10^{(3-4)} = 0.68 \cdot 10^{-1} = 6.8 \cdot 10^{-2}.$$

Notice how we don't stop until we have converted the answer into a number between 1 and 10 multiplied by a power of 10.

Exercise I.3

Evaluate, in scientific notation:
(a) $(1.5 \cdot 10^4)(7.5 \cdot 10^{-5})$; (b) $(4.3 \cdot 10^{-6})/(3.1 \cdot 10^{-10})$; (c) $(1.2 \cdot 10^{-5})(1.5 \cdot 10^3)/(9 \cdot 10^{-9})$.

To add or subtract numbers in scientific notation, you have first to rearrange the numbers so that all the powers of 10 are the same; then you can add or subtract the decimal parts, leaving the power of 10 alone. It's usually best if you first express all the numbers in terms of the *highest* power of 10. (In this connection, remember that , with negative powers of 10, smaller exponents means bigger numbers -- 10^{-3} is bigger than 10^{-5}.) You may, however, need to make a final adjustment if the combination of the decimal numbers is more than 10 or less than 1.

(Examples on the next page)

Examples: $2.1 \cdot 10^3 + 3.5 \cdot 10^5 = (0.021 + 3.5) \cdot 10^5 = 3.521 \cdot 10^5$.

$1.3 \cdot 10^8 - 8.4 \cdot 10^7 = (1.3 - 0.84) \cdot 10^8 = 0.46 \cdot 10^8 = 4.6 \cdot 10^7$.

$9.5 \cdot 10^{-11} + 9.8 \cdot 10^{-12} = (9.5 + 0.98) \cdot 10^{-11} = 10.48 \cdot 10^{-11} = 1.048 \cdot 10^{-10}$.

Exercise I.4

Evaluate: (a) $9.76 \cdot 10^9 + 7.5 \cdot 10^8$; (b) $1.25 \cdot 10^6 - 7.85 \cdot 10^5$; (c) $4.21 \cdot 10^{25} - 1.85 \cdot 10^{26}$; (d) $4.05 \cdot 10^{-19} - 10^{-20}$; (e) $(1.2 \cdot 10^{-19})(5.2 \cdot 10^{10} + 4 \cdot 10^9)$.

II. SIGNIFICANT FIGURES.

In the above examples and exercises in adding or subtracting numbers expressed in scientific notation, you will have noticed that you may end up with a large number of digits. But not all of these may be significant. For example, when we added $2.1 \cdot 10^3$ and $3.5 \cdot 10^5$, we got $3.521 \cdot 10^5$. However, each of the numbers being combined was given to only two-digit accuracy. (We are assuming that 3.5 means simply that the number is nearer to 3.5 than it is to 3.4 or 3.6. If it meant 3.500 then these extra zero digits should have been included.) This means that the $2.1 \cdot 10^3$ did not add anything significant to the bigger number, and we were not justified in giving more than two digits in the final answer, which should therefore have been given as just $3.5 \cdot 10^5$.

Note, however, that if we were asked to add, say, $8.6 \cdot 10^3$ to $3.5 \cdot 10^5$, the smaller number would make a significant contribution to the final answer. The straight addition would give us $3.586 \cdot 10^5$. Rounding this off to two digits would then give the answer as $3.6 \cdot 10^5$.

The general rules governing significant figures are:

1) The final answer should not contain more digits than are justified by the least accurate of the numbers being combined; but
2) Accuracy contained in the numbers being combined should not be sacrificed in the rounding-off process.

A few examples will help spell out these conditions (we'll ignore the powers-of-10 factors for this purpose):

Addition: $1.63 + 2.1789 + 0.96432 = 4.77422$, rounded to 4.77.

Subtraction: $113.2 - 1.43 = 111.77$, rounded to 111.8.

Multiplication: $(11.3)(0.43) = 4.859$, rounded to 4.9 (only two digits justified),

BUT $(11.3)(0.99) = 11.187$, rounded to 11.2 (three digits -- because rounding to two digits would imply only 10% accuracy, whereas each of the numbers being multiplied would justify 1%)

Division: $(1.30)/(0.43) = 3.023255814$, rounded to 3.0 (two digits),

BUT $(1.30)/(0.99) = 1.14141414141$, rounded to 1.14 (three digits justified).

Beware of Your Calculator! In the last two examples we used a pocket calculator to do the divisions. This automatically gave 10 digits in each answer. Most of those digits are INsignificant! Whenever you use your calculator for such a purpose, always ask yourself how many digits are justified and should be retained in the answer. Most of the calculations you will be doing will probably involve numbers with only a few significant digits. Get into the habit of cutting down your final answers to the proper size. As you can see, this situation will arise most importantly when you are doing divisions with your calculator. But watch out for surplus digits in multiplications as well.

Now let's turn to algebra proper.

III. LINEAR EQUATIONS

1) Equations in one variable

These scarcely need any discussion. There is one unknown, say x, and an equation that relates this to given numbers or constants. The only job is to tidy things up so as to solve for x explicitly.

Exercise III.1:

Solve for x: (a) $5\left(x+\frac{1}{4}\right) = 2x-\frac{1}{8}$; (b) $\frac{3}{x} - \frac{4}{5} = \frac{1}{x} + \frac{1}{3}$; (c) $3(ax+b) = 5bx + c$.

2) Simultaneous Equations in Two Variables:

However such equations are originally written, they can always be reduced to the following form:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2,$$

where the coefficients a, b, c may be positive or negative.

Two essentially equivalent methods can be used to obtain the solutions for x and y:

(a) Substitution: Use one of the equations (say the first) to get one of the unknowns (say y) in terms of x and known quantities. Substitute in the other equation to solve for the remaining unknown (x). Plug this value of x back into either of the initial equations to get y.

(b) Elimination: Multiply the original equations by factors that make the coefficient of one unknown (say y) the same in both. By subtraction, eliminate y; this leads at once to x:

Multiply 1st eq. by b_2: $\quad a_1b_2x + b_1b_2y = b_2c_1$;
Multiply 2nd eq. by b_1: $\quad a_2b_1x + b_2b_1y = b_1c_2$.
Subtract: $\quad (a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2$.

$$\therefore \qquad x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}.$$

Then solve for y as before. But the last equation above points to a special situation: If the denominator $(a_1b_2 - a_2b_1) = 0$, then x becomes indeterminate. (So, therefore, does y.)

[**Keep it neat!:** Good housekeeping in mathematics is very important. It will make it easier for you to check your work and it will help you to avoid errors. Notice how we put the "equals" signs underneath one another in the above analysis. Make a practice of doing this yourself. Concentrate on making this a habit, and don't throw it out of the window when you are taking a quiz or examination. You're bound to benefit from being neat and orderly.]

Exercise III.2.

Solve for both unknowns: (a) $2x - 3y = 4$; (b) $5a = b - 6$; (c) $(3/x) + (4/y) = 5$;
$3x - 2y = 5$. $2b = a + 4$ $(4/x) - (2/y) = 3$.

[In (c), resist the temptation to multiply both equations throughout by xy to clear the denominators. Just put $1/x = u$, $1/y = v$, and solve first for u and v, which are just as legitimate variables as x and y.]

IV. POLYNOMIALS

Much of what you do in algebra (and later, in calculus) will be based on a familiarity with expressions made up of a sum of terms like $10x^3$ or $(3.2)y^5$ -- in other words, sums of products of numbers called coefficients and powers of variables such as x. Such an expression is called a polynomial -- meaning simply something with many terms. Many important polynomials are made up of a set of terms each of which contains a different power of a single quantity, x. We can then write:

$$P(x) = a_o + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n,$$

where the quantities a_1, a_2, a_3, etc., are constant coefficients, labeled here to show which power of x they are associated with. We have written the combination as $P(x)$, to indicate that this combination, P, is a certain function of x if x is a mathematical variable. That is an aspect of polynomials which is important in calculus, but we won't expand on it in these algebra review notes.

A polynomial has a highest power of x in it; this is called the degree of the polynomial. Thus if the highest term is in x^4, we say that the polynomial is "a polynomial of degree four," or "a quartic." A polynomial of degree n can be written $P_n(x)$. For the most part, we shall not go beyond quadratics (degree 2).

The basis of many polynomials is a binomial (two-term) combination of two variables, of the form $(x + y)$, raised to an arbitrary power. The simplest examples of this are $(x + y)$ itself and $(x + y)^2 = x^2 + 2xy + y^2$. A binomial expansion is the result of multiplying out such an expression as $(x + y)^n$ into a polynomial.

If we want to add one polynomial to another, we simply add terms belonging to the same power of x.

Multiplying one polynomial with another is a bit more complicated, but again involves identifying all the terms that have the same power of x and adding the coefficients to make a single term of the form $a_n x^n$:

Example: $(x+2)(2x^2-3x+4)$

$$\begin{aligned}(x+2)(2x^2-3x+4) &= x(2x^2-3x+4) + 2(2x^2-3x+4)\\ &= (2x^3-3x^2+4x) + (4x^2-6x+8)\\ &= 2x^3 + (-3x^2+4x^2) + (4x-6x) + 8\\ &= 2x^3 + x^2 - 2x + 8.\end{aligned}$$

Exercise IV.1: Multiply $(3x^2+4x-5)$ by $(2x-1)$.

V. QUADRATIC EQUATIONS

A quadratic equation is a relationship that can be cast into the form:

$$ax^2 + bx + c = 0,$$

We can solve the equation through the process of completing the square. This technique is the basis of the general quadratic formula that you are very likely familiar with already:

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$$

We'll give a specific example and then derive the general formula.

1) Completing the Square: An Example.

Suppose we are asked to solve the equation:

$$x^2 - 6x - 4 = 0.$$

We recognize that the combination $(x^2 - 6x)$ is the first two terms of $(x-3)^2$. We just evaluate the complete square and see what is left over:

$$\begin{aligned}(x-3)^2 &= x^2 - 6x + 9\\ x^2 - 6x &= (x-3)^2 - 9\\ x^2 - 6x - 4 &= [(x-3)^2 - 9] - 4 = (x-3)^2 - 13.\end{aligned}$$

But $x^2 - 6x - 4 = 0$; $\quad (x-3)^2 - 13 = 0$, requiring $(x-3) = \pm\sqrt{13}$.

Thus the solution is: $\quad x = 3 \pm \sqrt{13}.$

[**An important point:** Notice that, in this example, our original quadratic expression reduces to a perfect square *minus* a certain number (13). When we set the whole expression equal to zero, this means that the perfect square is equal to a *positive* number, and we can proceed to take the square root of both sides. But if the quadratic had been a perfect square *plus* some number, n, we should have arrived at an equation of the form:

$$(x+p)^2 = -n.$$

We should then have been faced with taking the square root of a negative number. That would take us into the realm of "imaginary" numbers. That is beyond the scope of anything you need to consider for the present.]

2) <u>The General Quadratic Formula</u>

a) First rearrange the quadratic equation into the following form if it is not already in that form:

$$ax^2 + bx + c = 0.$$

b) Subtract the constant c from both sides:

$$ax^2 + bx = -c.$$

c) Divide through by the coefficient a:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

d) Complete the square of the left-hand side by adding $(b/2a)^2$. Add the same quantity to the right-hand side:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$$

e) Bring the right-hand side to a common denominator and take the square root of both sides:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}.$$

f) Subtract the constant $(b/2a)$ from both sides:

$$x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}.$$

And there you have the general quadratic formula.

<u>Notice that, if the solutions are to be real numbers, we must have $b^2 - 4ac > 0$.</u>

<u>Exercise V.1</u>

Solve these quadratic equations by completing the square. (If the coefficient of x^2 is not 1, you should follow the steps listed above in the derivation of the general formula):

(a) $x^2 - 4x - 12 = 0$; (b) $9x^2 - 6x - 1 = 0$; (c) $x + 2x^2 = 5/8$; (d) $(x+2)(2x-1) + 3(x+1) = 4$.

<u>Exercise V.2</u>.

Solve these quadratic equations by the general formula:

(a) $2x^2 = 9x - 8$; (b) Solve for t: $y = ut - \frac{1}{2}gt^2$; (c) $5x(x+2) = 2(1-x)$; (d) $0.2x^2 - 1.5x = 3$; (e) Solve for x: $x^2 - 2sx = 1 - 2s^2$.

VI. LONG DIVISION & FACTORING

Dividing one polynomial by another is very much like long division in arithmetic. We shall assume that the polynomial we are dividing into (the dividend) is of a higher degree than the one we are dividing by (the divisor). The reason for doing the division may be to find out if the higher-degree polynomial is exactly divisible by the lower-degree one; if it is, we have identified a way of *factoring* the higher-degree polynomial. We write both dividend and divisor in decreasing powers of x.

Example: Divide $4x^2 + 8x + 3$ by $2x + 1$.

$$\begin{array}{r@{\,}l} & \;2x + 3 \\ \cline{2-2} 2x + 1 &)\; 4x^2 + 8x + 3 \\ & \;\;\underline{4x^2 + 2x} \\ & \qquad\quad 6x + 3 \\ & \qquad\quad \underline{6x + 3} \\ & \qquad\qquad\quad \underline{0} \end{array}$$

It works! $(2x + 1)$ divides exactly into $4x^2 + 8x + 3$; the result is $(2x + 3)$, and so we have factored the quadratic into two linear factors.

Exercise VI.1: Divide $(5x^2 - 6x - 8)$ by $(x - 2)$.

Let's now change the above example of long division slightly, by making our initial polynomial $4x^2 + 8x + 5$. There will now be a remainder of 2.

The remainder 2 means that the quotient still has a fraction in it.

$$\frac{4x^2 + 8x + 5}{2x + 1} = 2x + 3 + \frac{2}{2x + 1}$$

This is analogous to the similar process with numbers; for instance:

$$\frac{423}{10} = 42 + \frac{3}{10}$$

With numbers we know that the remainder is always a number less than the divisor when dividing by 10; the possible remainders are 0, 1, ..., 9. For polynomials, the remainder is always of lower degree than the divisor. For example:

$$\frac{a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0}{x^2 + 1} = Q(x) + \frac{R(x)}{x^2 + 1}$$

The quotient $Q(x)$ has degree $3 = 5 - 2$ and the remainder $R(x)$ has degree at most 1. In other words, there are constants $b_0, b_1, b_2, b_3, c_0, c_1$ for which

$$Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0, \qquad R(x) = c_1x + c_0.$$

Exercise VI.2: Divide $(2x^3 + 2x^2 - 10x + 4)$ by $(x + 3)$. What is the degree of the remainder?

Exercise VI.3: If the polynomial $(x^5 + 4x^4 - 6x^3 + 5x^2 - 2x + 3)$ were divided by $(x^2 + 2)$ (Don't actually *do* it!), what would be the degree of the quotient and the degree of the remainder?

Exercise VI.4: What happens if you divide a polynomial of lower degree by one of higher degree?

Factoring is an important part of many calculations. But it is often hard to see which polynomial evenly (exactly) divides another. (In fact, factoring polynomials and numbers is an important problem for mathematicians and computer scientists.) Here is the only device that you need to know for now:

If $P(r) = 0$, then $(x - r)$ divides $P(x)$ evenly.

This is what we mean by saying that r is a root of $P(x)$.

Example $P(x) = x^2 + x - 2$.

$P(x)$ has the root -2, because $P(-2) = (-2)^2 + (-2) - 2 = 0$.

Then $P(x)$ is exactly divisible by $x - (-2) = x + 2$:

$$\begin{array}{r|l} & x - 1 \\ x + 2) & x^2 + x - 2 \\ & \underline{x^2 + 2x} \\ & \quad -x - 2 \\ & \quad \underline{-x - 2} \\ & \qquad 0 \end{array}$$

The quotient is $x - 1$. Therefore, $x^2 + x - 2 = (x + 2)(x - 1)$.
(The actual process of division here is of course just like that in the example at the beginning of the Section. But there we *gave* the value of the divisor, instead of looking for it from scratch.)

When dividing by a first order factor that divides evenly, you may save time by solving for coefficients using the following scheme:

$$\text{Put} \quad x^2 + x - 2 = (x + 2)(ax + b) = ax^2 + (2a + b)x + 2b.$$

Equating coefficients gives $a = 1, b = -1$. [But you can see in this case that $a = 1$ without writing down anything. Then you can solve for b in $x^2 + x - 2 = (x + 2)(x + b)$.]

Here is another approach to factoring: For polynomials $x^n + a_{n-1} x^{n-1} + ... + a_0$ with leading coefficient 1, and integers for the other coefficients, any integer root must be a divisor of the constant term a_0. Thus in the polynomial $x^2 + x - 2$ we have $a_0 = -2$, and there are four integers that might work: ± 1 and ± 2.

Finally, for quadratic equations the roots are always obtainable by the quadratic formula:

$$x^2 + x - 2 = 0 \;\rightarrow\; x = \frac{-1 \pm \sqrt{1^2 - 4(-2)}}{2} = \frac{-1 \pm 3}{2} = -2 \text{ or } +1.$$

In general, $ax^2 + bx + c = a(x - r_1)\ (x - r_2)$ where r_1 and r_2 are the roots. If the quadratic formula leads to the square root of a negative number then the quadratic has no real roots and no real factors. (If you allow the use of complex numbers as coefficients, then the quadratic factors as usual.)

[**Look out for easily factorable quadratics!** Be on the alert for the following:
a) Quadratics whose coefficients are small or simple enough for you to make a shrewd guess at the factorization;
b) Quadratics that are perfect squares;
c) Expressions of the form $a^2x^2 - b^2$, which can be immediately factored into $(ax + b)(ax - b)$.]

<u>Don't over-use the quadratic formula when factoring a polynomial like $x^2 + x - 2$</u>. Your first instinct should be to check ± 1 and ± 2 as roots. If you do use the quadratic formula, you should be prepared to <u>double</u> <u>check</u> your answer. Arithmetic errors can be as significant in a physical problem as the difference between going the right way or the wrong way down a one-way street.

<u>Exercise VI.5</u>:

Test whether the following quadratics can be factored, and find the factors if they exist:
(a) $8x^2 + 14x - 15$; (b) $2x^2 - 3x + 10$; (c) $9x^2 - 24ax + 16a^2$;
(d) $2x^2 + 10x - 56$; (e) $100x^4 - 10^{10}$. (This is of the 4th degree, but first put $x^2 = u$.)

<u>Exercise VI.6</u>:

Solve these quadratic equations by factoring:
(a) $x^2 - 5x + 6 = 0$; (b) $2x^2 - 5x - 12 = 0$; (c) $3x^2 - 5x = 0$; (d) $6x^2 - 7x - 20 = 0$.

VII. SOME TRICKS OF THE TRADE

1) Getting Rid of Radicals (Mathematical, not political)

Radicals are a nuisance if one is trying to solve an algebraic equation, and one usually wants to get rid of them. The way to get rid of a radical is of course to square it; but this only works if the radical stands by itself on one side of an equation. Thus if you have:

$\sqrt{(x-1)} + a = b$, you don't gain anything by squaring both sides.

This simply gives you: $(x-1) + 2a\sqrt{(x-1)} + a^2 = b^2$.

But if you first isolate the radical on the left-hand side, you have:

$$\sqrt{(x-1)} = b - a.$$

Then when you square you get: $x-1 = (b-a)^2$, and you are in business.

You may have to go through this routine more than once.

Example: Suppose you have $\sqrt{(2x-1)} - 1 = \sqrt{(x-1)}$.

Now you can't isolate both radicals -- and it doesn't help to put them together on one side of the equation. So you do the best you can, with one of the radicals isolated in the equation as it stands. Squaring, you get:

$$(2x-1) - 2\sqrt{(2x-1)} + 1 = x-1.$$

Now we can isolate the remaining radical:

$$2\sqrt{(2x-1)} = x+1.$$

Squaring again, and rearranging, gives us a quadratic equation:

$x^2 - 6x + 5 = 0$, leading to $x = 1$ or 5.

[**Warning!**: Squaring may introduce so-called extraneous roots, because, for example, if we started with $x = 3$ and squared it, we would have $x^2 = 9$. Taking the square root of *this* would appear to allow $x = -3$ as well as +3. So always check your final results to see if they fit the equation in its original form.]

Exercise VII.1.

Solve for x: (a) $\sqrt{(2x-1)} = x-2$ (Be careful!); (b) $\sqrt{(x-3)} = \sqrt{(2x-5)} - 1$.

2) Combining Fractions

a) Multiplying: $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$

b) Dividing: $\frac{(a/b)}{(c/d)} = \frac{a}{b}\frac{d}{c} = \frac{ad}{bc}$

(c) Adding: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

d) Subtracting: $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$

The quantities *a*, *b*, *c*, *d* may be ordinary numbers, but they may also be algebraic expressions. In either case, look first to see if they can be factored. In (a) and (b), this may lead to simple cancelation; in (c) and (d) it may enable you to find a least common denominator that is simpler than the product *bd*. In the latter case, convert each fraction to one with this common denominator and then add or subtract the numerators as required. This may involve less complexity than if you mechanically apply the expressions above.

Exercise VII.2.

Evaluate the following combinations of fractions, bringing the result to a single denominator in each case and canceling where possible:

(a) $\left(\frac{x-y}{x^2+y^2}\right)\left(\frac{y}{y-x}\right)$; (b) $\left(\frac{x^2y+y^3}{2x}\right)\Big/\left(\frac{2xy+y^2}{4x^3}\right)$; (c) $\frac{x}{4y^2z} + \frac{z^2}{8xy}$; (d) $\frac{4z}{xy^2} - \frac{2x}{y^3z} + \frac{z}{x^2y}$.

ANSWERS TO EXERCISES

Answer to exercises I.2 not provided.

Exercise I.1 (a) $8.0516 \cdot 10^4$; (b) $7.51 \cdot 10^{-2}$; (c) $3.52 \cdot 10^6$; (d) $8.1 \cdot 10^{-8}$

Exercise I.3 (a) 1.125; (b) $1.39 \cdot 10^4$; (c) $2 \cdot 10^6$

Exercise I.4 (a) $1.051 \cdot 10^{10}$; (b) $4.65 \cdot 10^5$; (c) $-1.429 \cdot 10^{26}$; (d) $3.95 \cdot 10^{-19}$;
(e) $6.72 \cdot 10^{-9}$

Exercise III.1 (a) $x = \frac{-11}{24}$; (b) $x = \frac{30}{17}$; (c) $x = \frac{(c-3b)}{(3a-5b)}$

Exercise III.2 (a) $x = 7/5,\ y = -2/5$; (b) $a = -8/9,\ b = 14/9$; (c) $x = 1,\ y = 2$

Exercise IV.1 (a) $6x^3 + 5x^2 - 14x + 5$

Exercise V.1 (a) 6,-2; (b) $\frac{1 \pm \sqrt{2}}{3}$; (c) $(-1 \pm \sqrt{6})/4$; (d) $(-3 \pm \sqrt{15})/2$

Exercise V.2 (a) $9 \pm \sqrt{17})/4$; (b) $(u \pm \sqrt{(u^2 - 2gy)})/g$; (c) $(-6 \pm \sqrt{46})/5$; (d) $(15 \pm \sqrt{465})/4$;
(e) $(s \pm \sqrt{(1-s^2)}$

Exercise V1.1 $5x + 4$

Exercise V1.2 The degree of the remainder is zero. (The remainder is $-2 = -2x^0$.)

Exericse V1.3 Degree of quotient: 3
Degree of remainder: 1 (at most)

Exercise V1.4 You can't do it by long division. We assumed that the divider had a lower degree than the dividend, in order for the long division to work. It's like trying to divide 2 by 7. So the answer is (again, without using long division!): quotient = 0; remainder = dividend.

Exercise V1.5 (a) $(4x-3)(2x+5)$; (b) No (complex roots); (c) $(3x-4a)^2$;

(d) $2(x+\frac{5}{2}-\frac{\sqrt{137}}{2})\,(x+\frac{5}{2}+\frac{\sqrt{137}}{2})$ (e) $100(x^2+10^4)(x+100)(x-100)$

Exercise V1.6 (a) $(x-2)(x-3)=0 \rightarrow 2,3$; (b) $(2x-3)(x+4)=0 \rightarrow 3/2,-4$;

(c) $x(3x-5)=0 \rightarrow \frac{5}{3},0$; (d) $(2x-5)(3x+4) \rightarrow \frac{5}{2},-\frac{4}{3}$

Exercise V11.1 (a) $x=5$ ($x=1$ doesn't work, since in the original equation the LHS is a radical (>0), and the RHS is $1\text{-}2=-1<0$); (b) $x=3$ or 7

Exercise V11.2 (a) $\frac{-y}{x^2+y^2}$ (b) $\frac{2x^2(x^2+y^2)}{2x+y}$ (c) $\frac{2x^2+yz^3}{8xy^2z}$ (d) $\frac{4xyz^2-2x^3+y^2z^2}{x^2y^3z}$

This module is based in large part on an earlier module prepared by the Department of Mathematics.