EXPONENTIALS & LOGARITHMS REVIEW MODULE

Introduction

Exponentials and logarithms could well have been included within the Algebra module, since they are basically just part of the business of dealing with powers of numbers or powers of algebraic quantities. But they have so much importance in their own right that it is convenient to give them a module of their own.

I. THE LAWS OF EXPONENTS

The concept of exponent begins with the multiplication of a given quantity a by itself an arbitrary number of times:

$$a^m = \underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \ldots}_{m \text{ times}}$$

On the left we have the product expressed in exponential notation, which is very compact and efficient. The expression above is in effect a definition of what we mean by an exponent.

If we multiply a by itself a total of $(m + n)$ times, we can of course write it as the m-fold product multiplied by the n-fold product:

$$\underbrace{a \cdot a \ldots}_{(m+n) \text{ times}} = (\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \ldots}_{m \text{ times}})(\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \ldots}_{n \text{ times}}).$$

This is far more easily expressed in exponential form, and gives us the first rule for dealing with quantities expressed in this way:

$$(a^m)(a^n) = a^{(m+n)}.$$

When two quantities, written as powers of a given number, are multiplied together, we add the exponents.

Note that if we took the quantity a^m and multiplied it by itself p times, this would be the same

as (i) raising the quantity a^m to the pth power;

or (ii) multiplying a by itself mp times.

Therefore:

$$(a^m)^p = a^{mp}.$$

When a quantity, written as a power of a given number, is itself raised to a certain power, the exponents multiply.

Be sure you keep clear in your mind the distinction between this formula and the previous one; the difference can be enormous if large powers are involved. Take, for instance, the following quantities:

$$(10^3)(10^6) = (10)^{(3+6)} = 10^9 \quad \text{-- one billion.}$$

But $(10^3)^6 = (10^3)(10^3)(10^3)(10^3)(10^3)(10^3) = 10^{18}$ -- a billion billion!

If we take a^m (the number a multiplied by itself m times), and *divide* it by a^n (a multiplied by itself n times), the result is a multiplied by itself $(m - n)$ times. Thus we have the rule for dividing one exponential by another:

$$\frac{a^m}{a^n} = a^{(m-n)}.$$

We can see from this that the reciprocal of any positive power of a is an equal negative power:

$$\frac{1}{a^n} = a^{-n}.$$

Also, if we put $m = n$ in the previous expression , $a^m/a^n = a^{(m-n)}$, the left-hand side is equal to 1. The right-hand side is a^0. Thus we have the result:

Any number (other than zero itself) raised to the power zero is equal to 1.

Exercise I.1. (No calculators!)
Evaluate: (a) $(2^4)(2^3)$; (b) $(10^2)(10^4)$; (c) $(10^2)^4$ [Compare with (b)]; (d) $(2^5)(3^{-3})$; (e) $(2^{-3})/(10^2)$.

Exercise I.2. (No calculators!)
Solve for x: (a) $x^5 = 32$; (b) $2^x = 1$; (c) $x^{-4} = 2$; (d) $10^x = 0.000\,000\,000\,1$.

NOTE: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

II. FRACTIONAL EXPONENTS, ETC.

Go back to the expression for raising a quantity to a certain power and then raising the resultant number to some other power:

$$(a^m)^p = a^{mp}.$$

If the product $mp = 1$, the right-hand side is just $a^1 = a$.

Suppose p is some specific integer, n. Then what the above equation says is that the parenthesis, raised to the nth power, is equal to a. But this means that the parenthesis is what we define as the nth root of a Also, since $mp = mn = 1$, we must have $m = 1/n$. Thus:

A fractional power corresponds to taking roots of numbers:

$$a^{(1/n)} = \sqrt[n]{a}.$$

We can proceed from this to consider a wider variety of exponentials:

1) Suppose we take the nth root of a and raise it to the power m. Expressed in exponential form, this can be written:

$$(\sqrt[n]{a})^m = a^{(m/n)}.$$

Thus we have a very convenient notation for writing any power of any root of a number.

2) The notation extends to negative powers also:

$$\frac{1}{(\sqrt[n]{a})^m} = a^{-(m/n)}.$$

[Considering exponents as formed from products or ratios of integers is enough for practical calculations, since these use only finite decimals, which are rational numbers. (For example, 1.032 = 1032/1000.) For non-rational values of exponents, limits are used. For example, $\sqrt{2} = 1.414...$, so $3^{\sqrt{2}}$ = limit of $3^1, 3^{1.4}, 3^{1.41}, 3^{1.414}, ...$]

Exercise II.1.

Simplify and evaluate: (a) $(2^8)^{1/2}$; (b) $(\sqrt{0.0016})^{-3}$; (c) $(8/27)^{2/3}$; (d) $(4^{1/3})^{9/2}$; (e) $(100)^{5/2}$.

EXPONENTIALS & ROOTS: SUMMARY	
$a^{(m+n)} = a^m a^n$	$a^{1/n} = \sqrt[n]{a}$ (*n* a positive integer)
$a^{-n} = 1/a^n$	$(a^m)^n = a^{mn}$
$a^0 = 1$	$a^m = a^n \Rightarrow m = n$ (if $a \neq 1$)

III. EXPONENTIALS AS FUNCTIONS

Using limits, as above, we can take the exponent to be any number we please, not just a rational number or fraction. In other words, we arrive at the concept that, in a quantity written as a^x, a can be any chosen number and x can be a continuous variable taking on any possible value from $-\infty$ to $+\infty$. Thus a^x becomes a continuous function of x.

$$y(x) = a^x.$$

This is an exponential function. A fixed (given) number (a) is raised to an arbitrary power (x). The quantity a is the base of the exponential function.

[Contrast this with the function x^n. Here a continuously variable number (x) is raised to a definite (given) power, n.]

Everyday life provides what is probably the most familiar example of an exponential function: the growth of a savings account with a fixed compound interest rate. If, for example, the interest is compounded annually at a rate of 5%, then the amount, A, in the account after n years, per dollar of initial deposit, is given by:

$$A(n) = (1.05)^n.$$

However, in this computer age, interest may vary daily. If we again assume a 5% annual rate, the daily interest earned by \$1 is 0.05/365, which is 0.00013698... The interest is added to your initial dollar, yielding 1.00013698 -- almost insignificantly different from 1. In that case, after, say, 200 days, your initial dollar will be worth $(1.00013698)^{200} = \$1.0277...$, a gain of almost 2.8 cents. (Check all this on your calculator.) If you deposit \$600, the calculation above is done for each dollar in the deposit, so you end up with a total sum of $600(1.00013698)^{200} = \$616.66$.

Exercise III.1.

You put \$200 into a savings account. (a) If the interest rate is 8%, compounded annually, how many years will it take for your account to reach \$250? (b) At the same annual rate, but compounded daily, what would be the balance in your account 500 days after the initial deposit?

The concept of the exponential function allows us to extend the range of quantities used as exponents. Besides being ordinary numbers, they can be expressions involving variables that can be manipulated in the same way as numbers.

Examples:

$$2^x\, 2^{-2x} = 2^{-x}; \qquad (10^{3x})^{1/x} = 10^3 = 1000.$$

Equations with the unknown in the exponent can be solved:

Example:

If $2^x = 4^{1/x}$, then $2^x = (2^2)^{1/x} = 2^{2/x}$, giving $x = 2/x$ and so $x = \pm\sqrt{2}$.

Watch the exponents! (This is an extension of what we said about integral exponents in Section I.) It is important not to get confused when you see a compound exponent. The notation should make things clear. Consider the following two cases:

If you see $(10^x)^2$, you should read this as $(10^x)(10^x) = 10^{2x}$.

But if you see 10^{x^2}, you should read it as $10^{(xx)} = (10^x)^x$.

If, for example, you put $x = 5$, the first expression is equal to 10^{10}, which is a big number; but the second expression is equal to 10^{25}, which is 15 orders of magnitude bigger.

Exercise III.2.

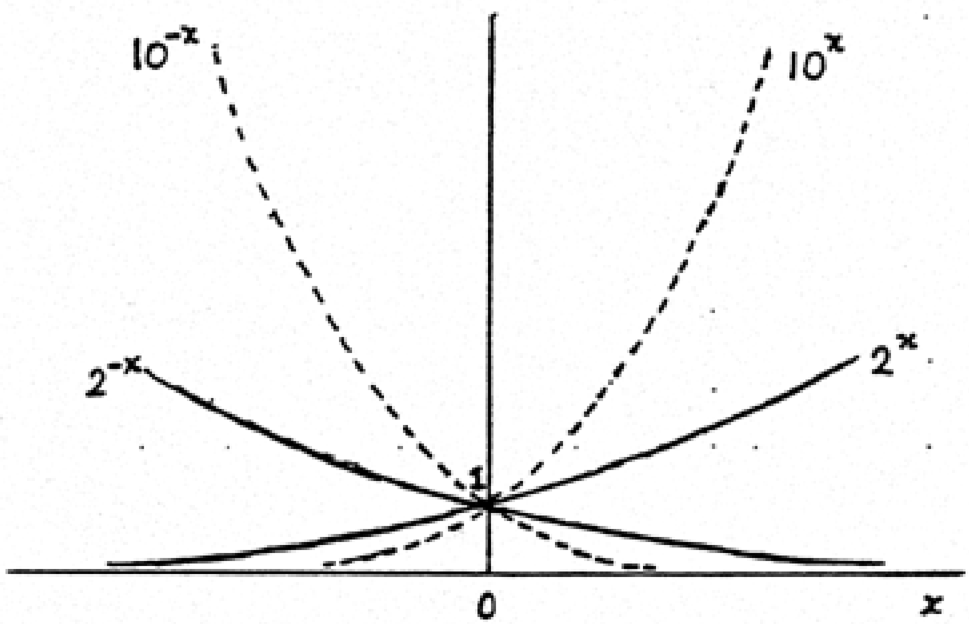
Combine and simplify: (a) $7^w\, 7^{2w}$; (b) $(3 \cdot 5^y)(5 \cdot 3^y)$; (c) $(2^4)^2$; (d) $16^a/2^b$.

IV. GRAPHS OF THE EXPONENTIAL FUNCTIONS

In mathematics and science, although the base a of the exponential function could in principle be any number, there are only three values of it that you will need to worry about for most purposes:

- $a = 2$: This is the basis of binary algebra, as used in computer science, etc.
- $a = 10$: This is the basis of most of our other scientific calculations.
- $a = e$: The symbol e stands for a very special irrational number whose first ten digits are 2.718281828. It is central to the use of exponential functions in calculus, but we will not consider it further here. (However, if you have already studied some calculus, you will very likely have met it.)

The graphs below show the general appearance of the exponential functions 2^x, 10^x and their reciprocals $(1/2)^x$ $(= 2^{-x})$ and $(1/10)^x$ $(= 10^{-x})$. All exponential functions are equal to 1 at $x = 0$. To describe an exponential that has some specific value other than 1 at $x = 0$, we simply put this value -- call it $y(0)$ -- in front as a multiplying or scaling factor.



<u>Notice that, with an exponential function, the factor of change for a given change of x is independent of where you start – the initial value (x_1) of x; it depends only on the difference ($x_2 - x_1$) between initial and final values:</u>

$$\text{If } y(x_1) = a^{x_1}, \text{ and } y(x_2) = a^{x_2}, \text{ then } \frac{y(x_2)}{y(x_1)} = a^{(x_2 - x_1)}.$$

Exponentials show up in all sorts of contexts. Here are a few examples:

<u>Positive Exponentials</u>

<u>Compound interest</u>, as already discussed:

$A(t) = A(0)(1 + c)^t$, where c is the compound interest rate per unit of time, and t is the time measured in those same units. (e.g., if the rate is 5% compounded annually, then $c = 0.05$ and t = time in years.)

Growth of a biological population:

A colony of bacteria, for example, grows by successive division, and may double in a few hours. One can put:

$$N(n) = N(0)(2)^n,$$ where n is the number of doubling times (τ)

since the population was equal to $N(0)$ – i.e., $n = t/\tau$.

Negative Exponentials

Radioactive decay:

This, like biological growth, can be described in terms of the time to produce a factor 2 of change -- but in this case a factor 2 decrease. This time is the half-life, $t_{1/2}$, and one has:

$$N(t) = N(0)\left(\frac{1}{2}\right)^{t/t_{1/2}} = N(0)(2)^{-t/t_{1/2}}.$$

Exercise IV.1.

(a) A colony of bacteria in a test tube doubles every hour. If there are 2500 bacteria in the tube when the experimenter leaves for lunch at 12:00 noon, how many are there when she comes back at 1:30 PM? How many are there at 5:00 PM?

(b) Find the half-life of a radioactive substance that has been left in a container for 6 days and decayed by a factor of 8?

V. LOGARITHMS

Definition: If b is a fixed positive number (other than 1!), and if two other positive numbers x and y are related by:

$$y = b^x,$$

we say that x is the logarithm of y to base b, and we write this relation in the form:

$$x = \log_b y.$$

This means that, for any (positive) number, N, and for any base, b, the following relation always holds:

$$\boxed{N = b^{\log_b N}}.$$

(The word "logarithm" is too much of a mouthful to use over and over again, so it is universally abbreviated as "log," not only in the above formula but also in speech.)

Below are some exercises based on the above definition:

Exercise V.1.

Find: (a) $\log_{10}1000$; (b) $\log_2 32$; (c) $\log_2(1/16)$; (d) $\log_b 1$; (e) $\log_3 27$; (f) $\log_2(-4)$.

That last set of exercises illustrates the following points:

1) Logarithms of numbers > 1 are positive;
2) Logarithms of positive numbers < 1 are negative;
3) The logarithm of 1 is zero in any base;
4) You can't have a logarithm of a negative number (until you get to complex numbers).

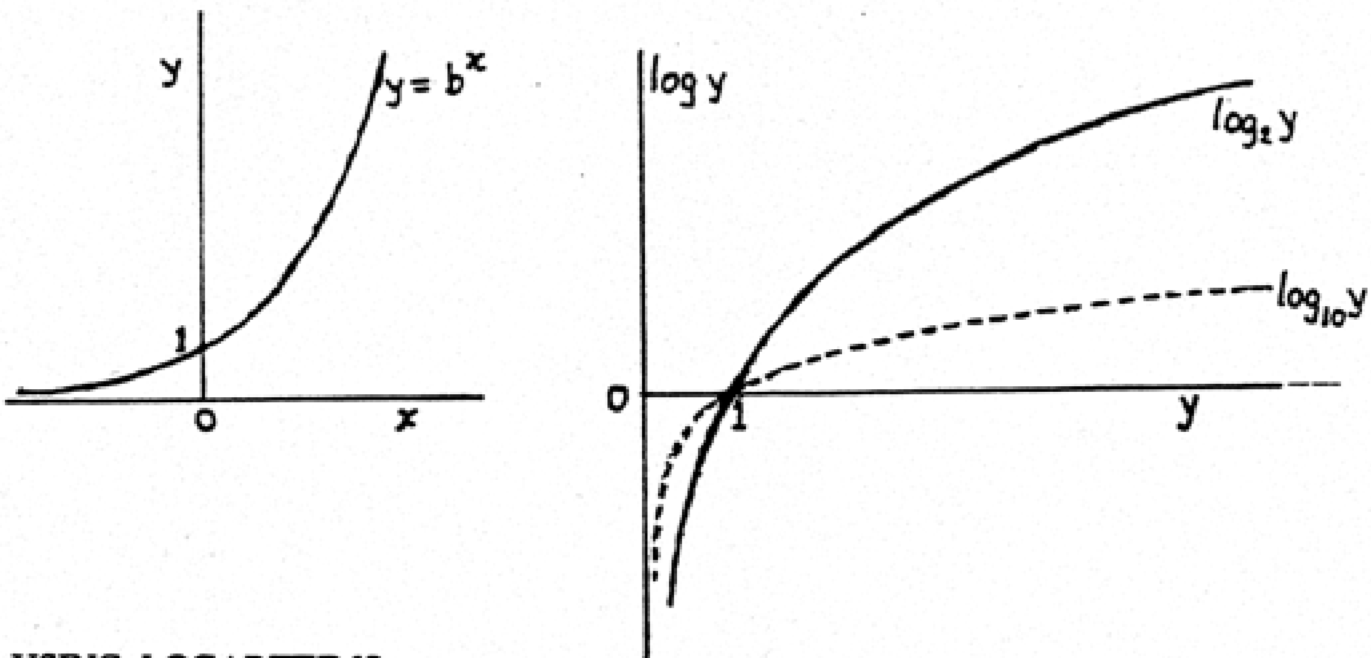
This last result follows from the definition of a logarithm. If $y = b^x$, with b a positive number, then y is positive; its smallest possible value is 0, when $x = -\infty$. In other words, $\log_b 0 = -\infty$, regardless of the value of the base b.

[**Logarithms to Base** e: Although we are not using the number e as a base of logarithms here, we should draw attention to the fact that the symbol for such logarithms is written as "ln" (for "natural logarithm") without any explicit identification of the base:

$$\ln x = \log_e x .$$

It is universally understood that any logarithm written this way is to base e. You will be meeting this constantly.]

The left-hand graph below is a reminder of the exponential dependence of y on x (= log y); the right-hand graph shows sketches of $\log_b y$ as a function of y for $b = 2$ and $b = 10$.



VI. USING LOGARITHMS

Since the log of a number is an exponent of some exponential, all we need to do to understand the properties of logarithms is to refer back to the properties of exponentials as summarized in Sections I and II of this review. These essential properties, which make logs so useful, are:

1) When two numbers in exponential form are multiplied together, the exponents add;

2) When one such number is divided by another, we subtract the exponents;

3) Since raising a number to a given power, p, means multiplying the number by itself p times, its exponent is multiplied by p.

Translated into the language of logs, these results become:

1') To multiply two numbers together, we add their logs:

$$\log_b(n_1 n_2) = \log_b n_1 + \log_b n_2.$$

2') To divide one number by another by another, we subtract their logs:

$$\log_b\left(\frac{n_1}{n_2}\right) = \log_b n_1 - \log_b n_2.$$

3') To raise a number to any power, we multiply its log by that power:

$$\log_b(n^p) = p\log_b n.$$

Of course, what we need as an answer to any such calculation is not just the *log* of the product, quotient or power, but the final numbers themselves. Therefore we have to go through the process of raising the base b to the power represented by the logarithm -- i.e., by the left-hand sides of the above equations. Remember:

$$N = b^{\log_b N}.$$

This is the process of finding the so-called antilogarithm of those quantities. Once upon a time, this had to be done by referring to tables of such antilogs -- just as the logs of the original numbers had to be read off from tables of logarithms. Nowadays, all we have to do is to push the appropriate buttons on our pocket calculators (using the INVERSE operation to get antilogs -- i.e., the final answers). But it's important to understand *in principle* what is involved. Below are some exercises to use these principles.

Exercise VI.1.

Given $\log_{10}2 = 0.301$, and $\log_{10}3 = 0.477$, find (a) $\log_{10}144$; (b) $\log_{10}(8/27)$; (c) $\log_{10}(2^{10})$.

Exercise VI.2.

Use your calculator to evaluate the antilogs (base 10) of the following logarithms: (a) 5; (b) 3.30103; (c) −0.69897 (d) the sum of (b) and (c); (e) the difference of (b) and (c). (This means evaluating 10^x where x is the given logarithm.)

ANSWERS TO EXERCISES

Exercise I.1. (a) 2^7; (b) 10^6; (c) 10^8; (d) 32/27; (e) $(1/8) \cdot (1/100) = 1/800$

Exercise I.2. (a) $x = 2$; (b) $x = 0$; (c) $x = 2^{-1/4}$; (d) $x = -10$

Exercise II.1 (a) $2^4 = 16$; (b) $25^3 = 15{,}625$; (c) 4/9; (d) $4^{3/2} = 8$; (e) $10^5 = 100{,}000$

Exercise III.1 (a) $n = \log_{1.08} 1.25 = 3$; or by trial-and-error: If $1.25 = (1.08)^n$,

try $n = 2$: $(1.08)^2 \cong 1.17 < 1.25$, $n = 3$: $(1.08)^3 \cong 1.25$, so $n = 3$ yrs.;
(b) \$223.16

Exercise III.2 (a) 7^{3w}; (b) 15^{y+1}; (c) 2^8; (d) 2^{4a-b}

Exercise IV.1 (a) 7,071; 80,000; (b) 2 days

Exercise V.1 (a) 3; (b) 5; (c) −4; (d) 0; (e) 3; (f) doesn't exist, by log definition

Exercise VI.1 (a) 2.158; (b) −0.528; (c) 3.01

Exercise VI.2 (a) 10,000; (b) 2,000; (c) 0.2; (d) 400; (e) 10,000

This module is based largely on an earlier module prepared by the MIT Mathematics Department.