GEOMETRY REVIEW MODULE

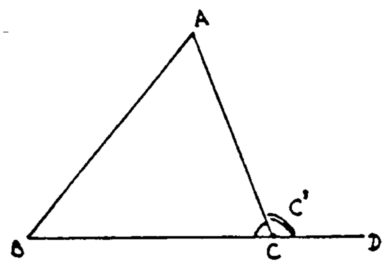
Introduction

This is a short module. It is not intended to be a full review of the theorems of Euclidean geometry. Its purpose is just to remind you of some of the chief results that will be of direct use to you in your work in science and in other areas of mathematics. In particular, the plane geometry in this module leads quickly into trigonometry, which is the subject of a separate module.

I. TRIANGLES

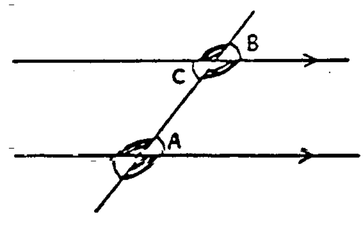
1) You will be familiar with the fact that, in Euclidean geometry, the angles of a triangle add up to 180°. We shall take this property of triangles as a given. It is worth remembering, though, that this is not a general result. It holds only for plane triangles. The angles of a triangle drawn on the surface of a sphere add up to *more* than 180°. The reason why that isn't important to us for most purposes is that the triangles we deal with usually have very small linear dimensions compared to the radius of the earth, so, for example, most surveying on the earth's surface can use Euclidean geometry with no significant error. (And then there are also other "non-Euclidean geometries" that obey different rules, which we'll certainly not deal with.) But here, this 180° property is a starting point:

In a triangle ABC, $\angle A + \angle B + \angle C = 180°$.



2) Another familiar property is that, if one side of a triangle is extended, as in the diagram here, the exterior angle C' is equal to the sum of the interior angles A and B. (A reminder of the proof: Since BCD is a straight line, $\angle C + \angle C' = 180°$, so $\angle C' = 180° - \angle C$. Then use the result above.)

3) Angles & Similar Triangles.



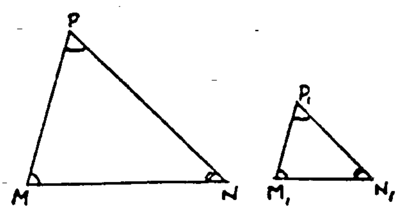
Some basic results of Euclidean geometry are very useful in analyzing geometrical situations in general and triangles in particular. Where one straight line cuts across two parallel straight lines:

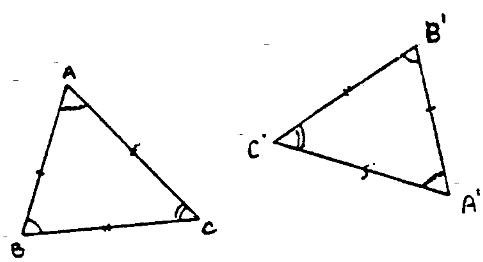
$\angle A = \angle B$ (because lines are parallel);

$\angle B = \angle C$ (opposite angles are equal);

$\angle A = \angle C$ (alternate interior angles are equal).

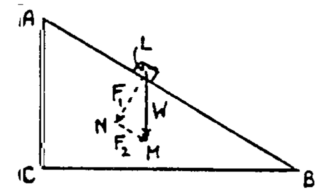
These find many applications in recognizing and relating *similar* triangles. It is easy to recognize similar triangles when they are placed side by side in the same orientation, like those below on the left. It may not be so easy if the triangles are in quite different orientations, like those on the right.





A common type of situation is the following mechanics problem.

Example:



A block sits on an inclined plane. It is acted on by the vertical force of gravity. In the solution of the problem, you want to *resolve* this gravitational force (the weight W) into components parallel and perpendicular to the plane.

The triangle LMN representing this analysis of the downward force W into two mutually perpendicular parts, F_1 and F_2, is similar to the triangle ABC composed of the inclined plane and its horizontal and vertical dimensions. One can therefore put:

$$F_1 : F_2 : W = BC : AC : AB.$$

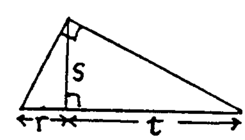
(This is more than an exercise in pure geometry, because we are comparing a geometrical triangle to a triangle of forces. But it is important and useful to know that this can be done.)

Exercise I.1.

Some artists used to use a *camera obscura* (literally just a "dark room") with a small hole in one wall to form an image of a distant scene on a screen inside the room. (Then all they had to do was to trace over the image!) Imagine a situation where the screen was 2 meters from the hole, forming an image of a building 10 meters high and 50 meters away. Draw a sketch showing rays of light passing to the screen from the top and the bottom of the building, and calculate the height of the image.

Exercise I.2.

Find three similar triangles in the figure opposite, and prove that $s^2 = rt$.



Exercise I.3.

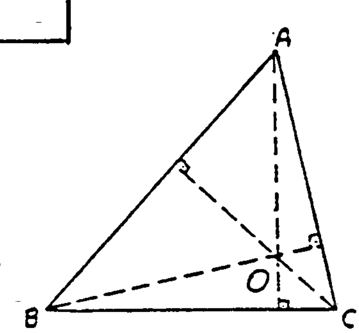
In the example of an inclined plane discussed above, find the forces F_1 and F_2 in terms of W if $\angle A = 30°$.

NOTE: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

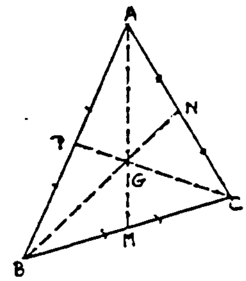
Orthocenters and Centroids.

In any triangle:

(a) The three perpendicular lines from the angles to the opposite sides (the altitudes) intersect at a single point. This is the orthocenter.

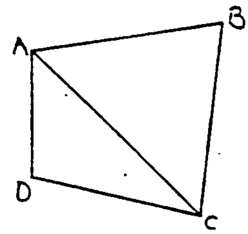


(b) The three lines from the angles to the mid-points of the opposite side (the medians) intersect at a single point.
This is the centroid.



The centroid is of particular significance physically. If a triangle is made from a sheet of material of uniform thickness, then the centroid is its balance point (its center of gravity), G. This can be understood by drawing lines parallel to each of the sides in turn. Suppose the figure is suspended from the corner A. Imagine the triangle divided up into strips parallel to the opposite side BC. Then the center of each strip lies on the line AM, and the triangle will be in balance, when hung from A, with AM vertical. Similarly for the other two medians. So, if suspended from the centroid, the triangle has no tendency to take up any particular orientation.

II. SOME OTHER PLANE FIGURES.

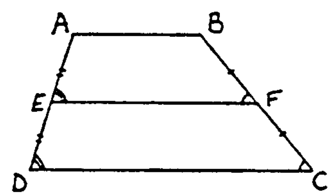


1) Quadrilaterals.

For any quadrilateral, the sum of the angles is 360°:

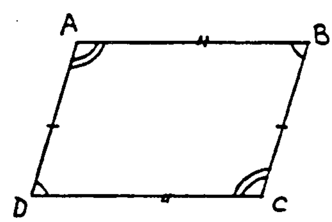
$$A + B + C + D = 360°.$$

(Think of it as two triangles, ABC and ACD.)



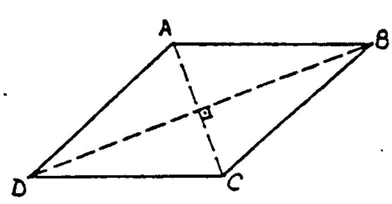
Some special quadrilaterals:

(a) Trapezoid, with AB parallel to CD. Its width at half altitude is equal to $(AB + CD)/2$.



(b) Parallelogram, with AB parallel to CD, and AD parallel to BC.
Other properties:

$$\angle A = \angle C; \qquad \angle B = \angle D.$$
$$AB = DC, \qquad AD = BC.$$



(c) Rhombus, a parallelogram with all sides equal:

$$AB = BC = CD = DA.$$

Its diagonals are perpendicular.

2) Other Polygons.

The literal meaning of "polygon" is "many-cornered," but we usually think in terms of the number of sides. Since the number of angles equals the number of sides, n, you can't go wrong. It's useful to know the names of a few polygons beyond the quadrilateral:

$n = 5$:	Pentagon;	$n = 8$	Octagon;
$n = 6$:	Hexagon;	$n = 10$	Decagon.
$n = 7$:	Heptagon;	$n = 12$:	Dodecagon.

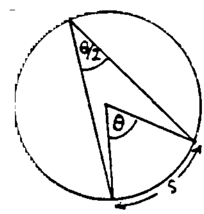
(The ones omitted are seldom met.)

There is a general formula for the sum of all the angles of a polygon with n sides or corners:

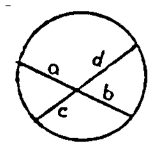
$$\text{Angle Sum } (S) = (n - 2) \times 180^\circ.$$

III. THE ALL-MIGHTY CIRCLE

Some properties worth knowing about -- even if you don't memorize them:



1) Given any arc of the circle, the angle it subtends at the center is twice the angle it subtends at the circumference. (Special case: A semicircle subtends 180° at the center and a right angle at the circumference.)



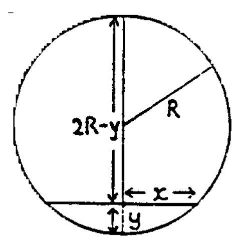
2) Intersecting chords: The products of their separate parts are equal:

$$a \cdot b = c \cdot d.$$

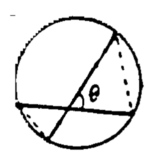
(Special case: A diameter and a chord perpendicular to it: With the figure labeled as shown,

$$y(2R - y) = x^2.$$

If $y \ll R$, then $y \approx x^2/2R$. This means that a small part of a circle has almost the same shape as a parabola. That is important in many mathematical and physical approximations. Admittedly, this is analytic geometry, not plain geometry!)



3) The figures formed by any two intersecting chords, not necessarily passing through the center of the circle, are geometrically similar.



IV. LENGTHS, AREAS & VOLUMES.

There are many useful formulas for the perimeters, areas and volumes of geometrical figures. Some of them you should definitely memorize. But before you look at the tabulation on the next page, consider the following statements concerning what are called the dimensions of such quantities:

Any perimeter must have the dimensions of length. That is, it must be expressed in terms of length to the first power only.

Any area must have the dimensions of (length)^2.

Any volume must have the dimension of (length)^3.

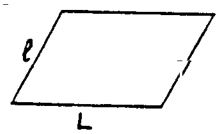
For example, if you are trying to remember the formula for the area of a circle, consider that it couldn't *possibly* be $2\pi r$. Whatever else there may be, it's got to have r to the 2nd power. Guessing that it's $2\pi r^2$ (as against the correct formula πr^2) is an error you shouldn't make -- but it's less serious than using a formula that could only represent a length.

Likewise, the circumference of a circle could not possibly be given by πr^2 or $2\pi r^2$.

FORMULAS

Make a special effort to memorize the formulas marked with an asterisk (*) People very often get them wrong!

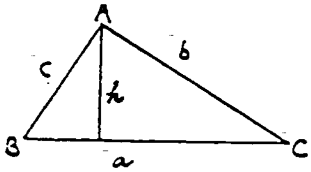
1) Perimeters



Rectangle or Parallelogram: $2(L + l)$.

*Circle: $2\pi R$

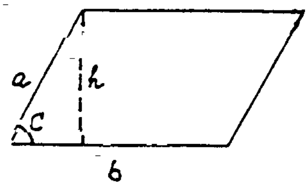
Any other straight-sided figure: Just add up the lengths.



2) Areas

Triangle: $\frac{1}{2}bh = \frac{1}{2}ab\sin C$ = etc. (cyclically)

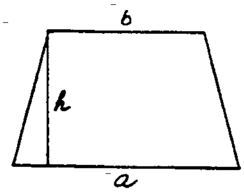
Parallelogram: $bh = ab\sin C$.



Trapezoid: $\frac{1}{2}h(a + b)$.

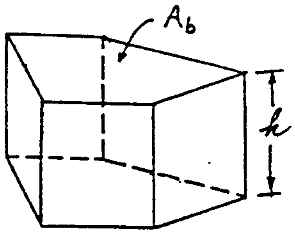
*Circle: πR^2.

*Cylinder: $2\pi Rh$ (sides) + $2\pi R^2$ (ends).



Prism: $P_b h + 2A_b$.
(P_b = perimeter of base;
A_b = area of base;
h = height.)

*Sphere: $4\pi R^2$.



Volumes

Rectangular Parallepiped: $abc = A_b h$.

Pyramid or Cone: $\frac{1}{3}A_b h = \frac{1}{3}\pi R^2 h$ (for circular cone)

*Sphere: $\frac{4\pi}{3}R^3$.

Exercise IV.1.

Prove that the area of a flat circular ring or washer, with inner radius r_1 and outer radius r_2, is equal to $2\pi r_{\text{ave.}} \Delta r$, where $r_{\text{ave.}}$ is the average radius $(r_1 + r_2)/2$, and Δr is the width $(r_2 - r_1)$.

Exercise IV.2.

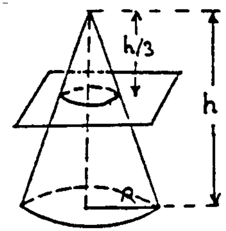
Given that a sphere has area 64π, what is its volume?

Exercise IV.3.

A cube of edge-length L has a sphere just fitting inside it (i.e., the diameter of the sphere is equal to L). Calculate the ratios of (a) the surface areas, and (b) the volumes of these two figures.

Exercise IV.4.

A plane, parallel to the base of a circular cone of height h and of radius R at the base, cuts across it at a distance of $h/3$ from the top.
Calculate the ratio of the volumes of the small cone and the original cone.

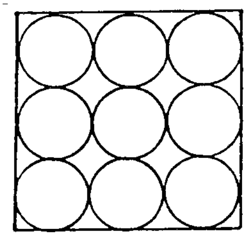


Exercise IV.5.

A cube of side a is packed full with small spheres of diameter a/n (n = positive integer) as shown in the diagram.

Consider the total volume of the spheres. Do you think it increases or decreases as n gets bigger?

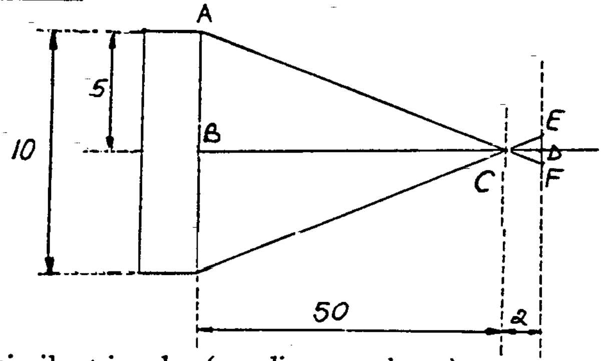
Given two equal-size and equal-price boxes of mothballs, should you buy the one with the larger mothballs or the one with the smaller mothballs?



ANSWERS TO EXERCISES

Exercise I.1

Drawing not to scale.



Find similar triangles (see diagram above):

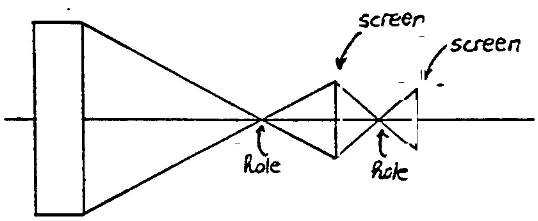
$$\Delta ABC \sim \Delta CDF \rightarrow \frac{AB}{BC} = \frac{DF}{CD}$$

or, using the distances given,

$$\frac{5}{50} = \frac{DF}{2} \rightarrow DF = \frac{1}{5}$$

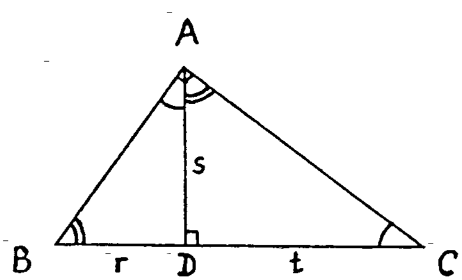
Then the height FE of the image is $FE = 2 \cdot DF = \frac{2}{5}\text{m} = 0.4\text{m}$.

Note: If the artists used only one hole in the wall, they would trace an inverted image on the screen. (The rays passing through the hole from the top and the bottom of the building end up on the screen at points F and E, respectively).



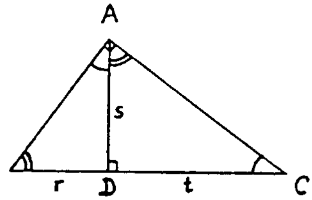
In order to get the image right side up we would need two holes and two screens (see diagram on left.)

Exercise I.2



$$\Delta ABC \sim \Delta BDA \rightarrow \frac{AB}{AC} = \frac{BD}{AD} \quad (1)$$

$$\Delta ABC \sim \Delta ADC \rightarrow \frac{AB}{AC} = \frac{AD}{DC} \quad (2)$$



Using both (1) and (2), we have:

$$\frac{BD}{AD} = \frac{AD}{DC}, \text{ or } \frac{r}{s} = \frac{s}{t} \rightarrow s^2 = rt.$$

[Solution without using similar triangles, but using Pythagoras's theorem instead:

$$AB^2 = BD^2 + AD^2 \rightarrow AB^2 = r^2 + s^2 \qquad (1)$$

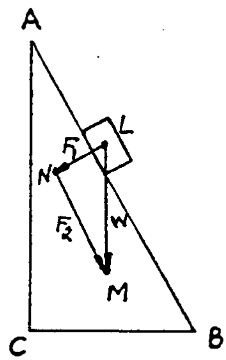
$$AC^2 = AD^2 + DC^2 \rightarrow AC^2 = s^2 + t^2 \qquad (2)$$

But $BC^2 = AB^2 + AC^2$, and $BC = r + t$. By substitution (see (1) and (2) we obtain:

$$BC^2 = r^2 + s^2 + s^2 + t^2$$
$$(r+t)^2 = r^2 + 2s^2 + t^2$$
$$r^2 + 2rt + t^2 = r^2 + 2s^2 + t^2$$

or $2rt = 2s^2$, which gives us the answer, $rt = s^2$.

<u>Exercise I.3</u>



$$\Delta ABC \sim \Delta LMN \rightarrow \frac{AB}{LM} = \frac{AC}{MN} = \frac{BC}{LN}, \text{ or } \frac{AB}{W} = \frac{AC}{F_2} = \frac{BC}{F_1},$$

thus $\frac{BC}{AB} = \frac{F_1}{W}$ and $\frac{AC}{AB} = \frac{F_2}{W}$ (1)

$\angle A = 30°$. Using the known relationships in a 30° - 60° - 90° triangle, (mentioned in the Trigonometry Module),

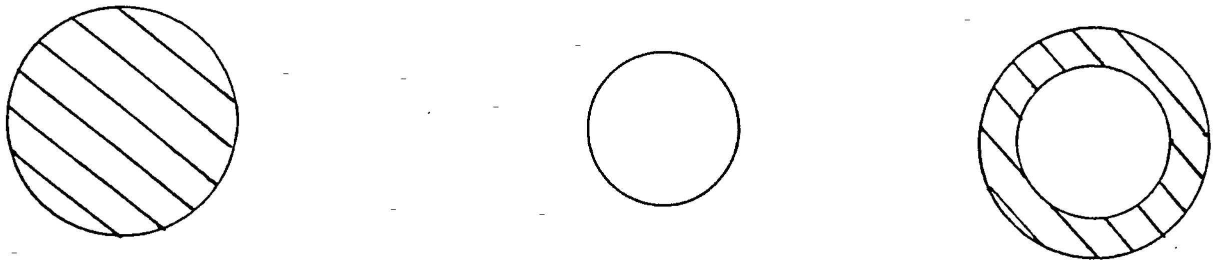
$$\frac{BC}{AB} = \frac{1}{2}, \quad \frac{AC}{AB} = \frac{\sqrt{3}}{2}.$$

Using the two equalities above and (1), we obtain:

$$\frac{BC}{AB} = \frac{1}{2} = \frac{F_1}{W} \rightarrow F_1 = \frac{W}{2}$$

$$\frac{AC}{AB} = \frac{\sqrt{3}}{2} = \frac{F_2}{W} \rightarrow F_2 = \frac{W\sqrt{3}}{2}.$$

Exercise IV.I



$A_{outer} = \pi r_2^{\,2}$ $\qquad$ $A_{inner} = \pi r_1^{\,2}$

The washer is what is left of the outer circle, when the inner one is taken away.

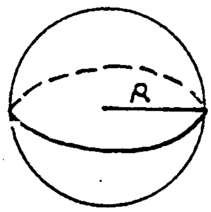
Its area will be:

$$A_w = \pi r_2^{\,2} - \pi r_1^{\,2}.$$

Using a few algebraic manipulations,

$$A_w = \pi(r_2^2 - r_1^2) = \pi(r_2 + r_1)(r_2 - r_1) = 2 \cdot \pi \cdot \frac{(r_2 + r_1)}{2}(r_2 - r_1) = 2\pi r_{ave}\,\Delta r.$$

Exercise IV.2

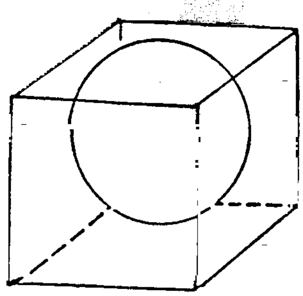


$$A_{sphere} = 4\pi R^2 = 64\pi$$

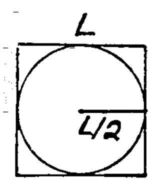
$$R^2 = 16,\ R = 4$$

$$V_{sphere} = \frac{4\pi R^3}{3} = \frac{4\pi \cdot 64}{3} = \frac{256\pi}{3}.$$

Exercise IV.3



The radius of the sphere is $R = \frac{L}{2}$.



a) $A_{\text{cube}} = 6 \cdot L^2$

$$A_{\text{sphere}} = 4\pi R^2 = 4\pi\left(\frac{L}{2}\right)^2 = \pi L^2$$

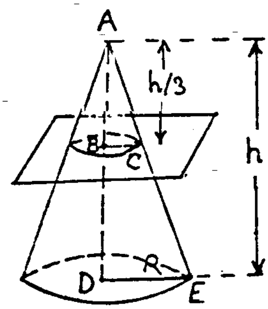
$$\frac{A_{\text{cube}}}{A_{\text{sphere}}} = \frac{6L^2}{\pi L^2} = \frac{6}{\pi} \cong 1.91$$

b) $V_{\text{cube}} = L^3$

$$V_{\text{sphere}} = \frac{4\pi R^3}{3} = \frac{4\pi}{3} \cdot \left(\frac{L}{2}\right)^3 = \frac{\pi L^3}{6}$$

$$\frac{V_{\text{cube}}}{V_{\text{sphere}}} = L^3 / \left(\frac{\pi L^3}{6}\right) = \frac{6}{\pi} \cong 1.91$$

<u>Exercise IV.4</u>



$$\Delta ABC \sim \Delta ADE \rightarrow \frac{AB}{AD} = \frac{BC}{DE}, \text{ or } \frac{h/3}{h} = \frac{BC}{R} \rightarrow BC = \frac{R}{3}.$$

$$V_{\text{sm. cone}} = \frac{1}{3}\pi \cdot (BC)^2 \cdot AB = \frac{1}{3}\pi\left(\frac{R}{3}\right)^2 \cdot \left(\frac{h}{3}\right) = \frac{\pi R^2 h}{81}$$

$$V_{\text{lg. cone}} = \frac{1}{3}\pi (DE)^2 \cdot AD = \frac{1}{3}\pi R^2 h.$$

$$\frac{V_{\text{sm. cone}}}{V_{\text{lg. cone}}} = \frac{\pi R^2 h}{81} \cdot \frac{1}{\pi R^2 h} = \frac{1}{27}$$

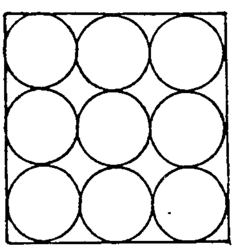
3

[N.B.: no units, since it is a <u>ratio</u> of two quantities with the same dimension, (length)^3.]

Another Method (Dimensional Analysis): Since volume has dimension (length)^3, the formula for V must be $V = ch^3$ for some constant c related to the angle at the vertex. Hence,

$$\frac{V_{\text{sm. cone}}}{V_{\text{lg. cone}}} = \frac{c(h/3)^3}{ch^3} = \frac{ch^3/27}{ch^3} = \frac{1}{27}.$$

<u>Exercise IV.5</u>



Since the diameter of the small spheres is a/n, the radius will be

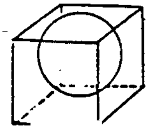
$$R = \frac{a}{2n}.$$

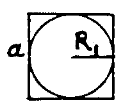
Considering the three-fold symmetry of the sphere and cube, it should be clear that we are dealing with n^3 small spheres packed inside the cube.

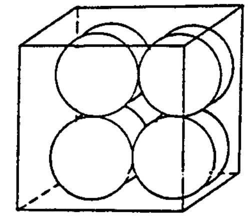
Take first just one big mothball (sphere) inside the cube.

$$n = 1; \quad n^3 = 1$$

$$R_1 = \frac{a}{2};$$







The next number of spheres we can pack is 8:
with $n = 2$, $n^3 = 8$

$$R_2 = \frac{a}{2 \cdot 2} = \frac{a}{4} \text{ (check it for yourself).}$$

and so on -- you can continue with $n = 3$, packing $n^3 = 27$ spheres of radius $R_3 = \frac{a}{6}$, etc.]

The volume of one sphere is: $V = \frac{4\pi R^3}{3} = \frac{4\pi}{3} \cdot \left(\frac{a}{2n}\right)^3 = \frac{\pi a^3}{6n^3}$.

But since we have n^3 spheres, the total volume of the spheres is

$$V_{tot} = n^3 \cdot V = n^3 \cdot \frac{\pi a^3}{6n^3} = \frac{\pi a^3}{6}.$$

So the final result is independent of n; no matter which you buy, the total mass of the mothballs – assuming constant density – is the same. (This result is exactly the same as in Exercise IV.3 b.)

The ratio of the volume of the inscribed sphere to the volume of the cube is $\pi/6$ regardless of the sidelength L.)

This module is based in large part on an earlier module prepared by the Department of Mathematics.