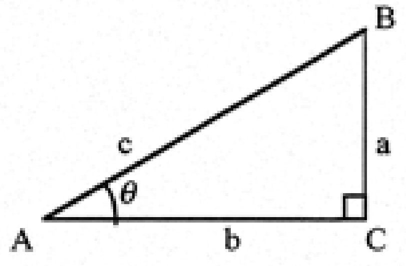
TRIGONOMETRY REVIEW MODULE

Introduction

As you probably know, trigonometry is just "the measurement of triangles", and that is how it got started, in connection with surveying the earth and the universe. But it has become an essential part of the language of mathematics, physics and engineering.

I. RIGHT TRIANGLES

The simplest place to begin this review is with right triangles. We just have an angle θ ($0° < \theta < 90°$), and the lengths of the sides a, b, c. With this labeling of the sides, we have:



- a is the side opposite to θ;
- b is the side adjacent to θ;
- c is the hypotenuse (literally the "stretched side").

From these we construct the three primary trigonometric functions --sine, cosine and tangent:

$$\sin\theta = \frac{a}{c};\ \cos\theta = \frac{b}{c};\ \tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta}.$$

Some people remember these through a mnemonic trick -- the nonsense word **SOHCAHTOA**:

$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}};\qquad \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}};\qquad \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}.$$

Perhaps you yourself learned this. But you'll be much better off if you simply know these relations as a sort of reflex and don't have to think about which ratio is which.

You will also need to be familiar with the reciprocals of these functions--

cosecant=1/sine; secant=1/cosine; cotangent =1/tangent:

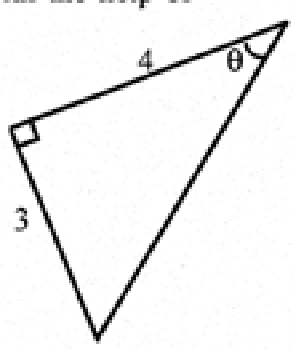
$$\csc\theta = \frac{1}{\sin\theta} = \frac{c}{a};\ \sec\theta = \frac{1}{\cos\theta} = \frac{c}{b};\ \cot\theta = \frac{1}{\tan\theta} = \frac{b}{a}.$$

If we wish, we can of course express the hypotenuse c in terms of a and b with the help of Pythagoras' Theorem:

$$c^2 = a^2 + b^2, \text{ so } c = \sqrt{a^2 + b^2} = (a^2 + b^2)^{1/2}.$$

Exercise I.1

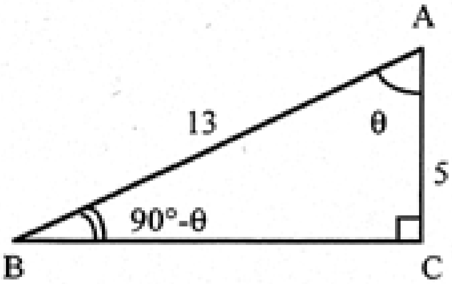
Cover up the formulas above. Then find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$, $\cot\theta$, in the triangle shown here. (We've deliberately drawn it in a non-standard orientation; you need to be able to handle that sort of thing.)



NOTE: The answers to the exercises are all collected together at the end of this module. We have tried to eliminate errors, but if you find anything that you think needs to be corrected, please write to us.

Exercise I.2

In this triangle, find: sin $(90°\text{-}\theta)$;
sinθ + cos $(90°\text{-}\theta)$;
tanθ + cot $(90°\text{-}\theta)$
secθ + csc $(90°\text{-}\theta)$;



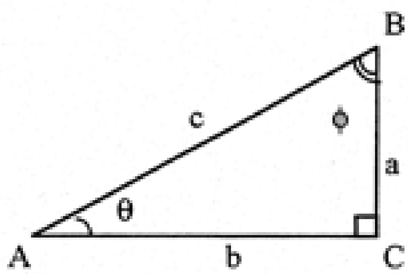
[Note: $(90°\text{-}\theta)$ is a perfectly valid name for the angle at *B*, though for some purposes we might want to call it, say, β for simplicity. But the important thing here is just to get the relation of the sines and cosines, etc., straight. Here, θ is what we might call the primary angle, $(90°\text{-}\theta)$ is the co-angle (complementary angle). The above exercise is designed to make the point that the sine, tangent and secant of the angle θ have the same values as the co-sine, co-tangent and co-secant of the co-angle $(90°\text{-}\theta)$ -- and *vice versa*.]

Exercise I.3

Sorry, folks. No picture this time. You draw the triangle.
If *A* is an acute angle and sin A = 7/25, find all the other trigonometric functions of the angle *A*.

IMPORTANT!

It's not enough to know the *definitions* of the various trigonometric functions. You also need to be able to use them to find the length of any side of a right triangle in terms of any other side and one of the angles. That is, in the triangle *ABC*, in which *C* is the right angle, you should be familiar with the following relationships:



$a = b \tan\theta = c \sin\theta = c \cos\phi = b \cot\phi$;
$b = a \cot\theta = c \cos\theta = c \sin\phi = a \tan\phi$;
$c = a \csc\theta = b \sec\theta = a \sec\phi = b \csc\phi$.

In particular, the relations $a = c \sin\theta$, and $b = c \cos\theta$, correspond to the process of breaking up a linear displacement or a force into two perpendicular components. This operation is one that you will need to perform over and over again in physics problems. Learn it now so that you have it ready for instant use later.

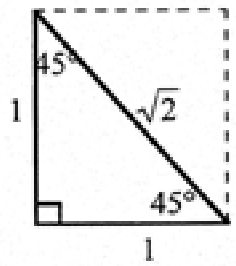
Exercise I.4

In a right triangle labeled as above, find:
(a) *BC*, if $A = 20°$, $AB = 5$;
(b) *AC*, if $B = 40°$, $BC = 8$;
(c) *AB*, if $A = 53°$, $AC = 6$.

II. SOME SPECIAL TRIANGLES

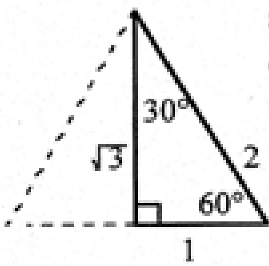
We've already used some special triangles in Section I, above -- right triangles in which all three sides can be expressed as integers: (3, 4, 5), (5, 12, 13), and (7, 24, 25). It's very convenient to be familiar with such triangles, for which Pythagoras' Theorem becomes just a relation between the squares of the natural numbers. (You might like to amuse yourself hunting for more examples.) And people who set examinations are fond of using such triangles, to save work for the students who take the exams and for the people who grade them. So there can be a very practical advantage in knowing them!

And there are some other special triangles that you should know inside out. Look at the two triangles below. No doubt you're familiar with both of them. The first half of a square and the second is half of an equilateral triangle. Their angles and principal trigonometric functions are as shown.



$\sin 45° = 1/\sqrt{2} = \cos 45°$

$\tan 45° = 1$



$\sin 30° = 1/2 = \cos 60°$

$\cos 30° = \sqrt{3}/2 = \sin 60°$

$\tan 30° = 1/\sqrt{3} = \cot 60°$

These triangles, too, often show up in quizzes and examinations. You will very likely be expected to know them in tests where calculators are not allowed. More importantly, though, they should become part of your "stock-in-trade" of known numerical values that you can use for problem-solving. Getting an *approximate* answer to a problem can often be very useful. You might, for example, be doing a calculation in which the cosine of 58.8° shows up. Maybe you don't have your calculator with you -- or maybe it has decided to break down. If you know that cos 60° = 1/2, you can use this as a good approximation to the value you really want. Also, since (for angles between 0° and 90°) the cosine gets smaller as the angle gets bigger, you will know that the true value of cos 58° is a *bigger* than 1/2; that can be very useful information, too.

Exercise II.1

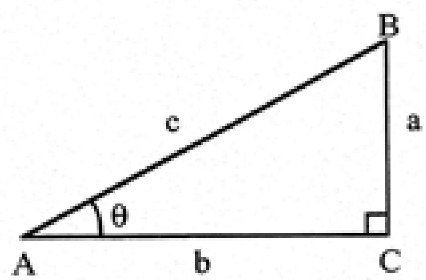
Take a separate sheet of paper, draw the 45°/45°/90° and 30°/60°/90° triangles, and write down the values of all the trig functions you have learned. Convert the values to decimal form also.

Exercise II.2

Given: this triangle, with $C = 90°$.

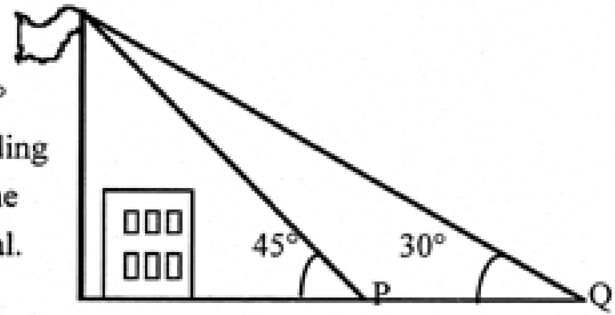
Find all angles and lengths of the sides if:

a) $A = 30°$, $b = 6$
b) $a = 13$, $b = 13$.
c) $B = 30°$, $c = 10$.
d) $A = 45°$, $c = 12$.
e) $c = 1$, $b = \sqrt{2}$
f) $c = 4$, $b = 4$.



Exercise II. 3

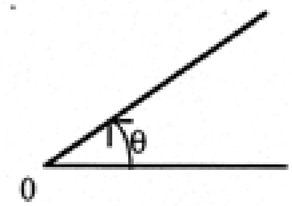
A tall flagpole stands behind a building. Standing at point *P*, you observe that you must look up at an angle of 45° to see the top of the pole. You then walk away from the building through a distance of 10 meters to point *Q*. From here the line to the top of the pole makes an angle of 30° to the horizontal. How high is the top of the pole above eye level?



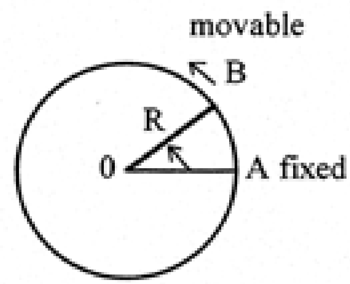
[If you don't see how to approach this problem, take a peek at the solution -- just the diagram at first.]

III. RADIAN ANGLE MEASURE

A given angle θ is uniquely defined by the intersection of two straight lines. But the actual *measure* of the angle can be expressed in different ways. For most practical purposes (e.g., navigation) we use the division of the full circle into 360 degrees, and measure angles in terms of degrees, minutes and seconds of arc. But in mathematics and physics, angles are usually expressed in radians. Radians are much more useful than degrees when you are studying functions, graphs, and such things as periodic motion. This is because radians simplify all calculus formulas for trig functions. The price we pay for simplicity is that we need to introduce the fundamental constant Π. But that is worth understanding anyway.



Imagine a circle, with an angle formed between two radii. Suppose that one of the radii (the horizontal one) is fixed, and that the other one is free to rotate about the center so as to define any size of angle we please. To make things simple, choose the circle to have radius $R = 1$ (inch, centimeter, whatever -- it doesn't matter). We can take some flexible material (string or thread) and cut off unit length of it (the same units as we've used for *R* itself). We place one end of the string at the end *A* of the fixed radius, and fit the string around the contour of the circle. If we now place the end *B* of the movable radius at the other end of the string, the angle between the two radii is 1 radian (Mnemonic: 1 radian = 1 radius-angle).



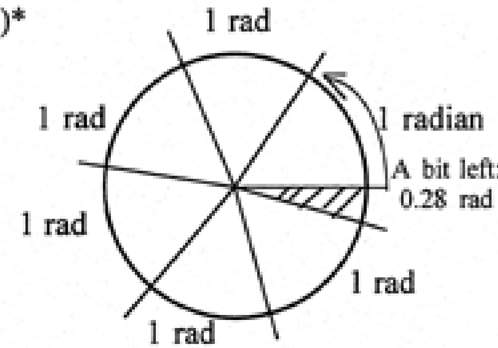
To put it formally:

A radian is the measure of the angle that cuts off an arc of length 1 on a unit circle.

Another way of saying this is that an arc of unit length subtends an angle of 1 radian at the center of a unit circle.

[If you don't like unit circles, you can say that an angle of one radian cuts off an arc equal in length to the radius on a circle of *any* radius.]

If we take a length of string equal to the radius and go all the way around the circle, we find that we can fit in 6 lengths plus a bit more (about 0.28 of the radius)* For one half of the circle, the arc length is equal (to an accuracy of 2 decimal places) to 3.14 radii. Surprise: $3.14 \approx \Pi$! (No, not a surprise. This is how Π defined.) So we say there are Π radians in 180°, and we have:



2Π radians is equivalent to a 360° rotation;

Π radians is equivalent to a 180° rotation.

The abbreviation "rad" is almost always used when we write the value of an angle in radians.

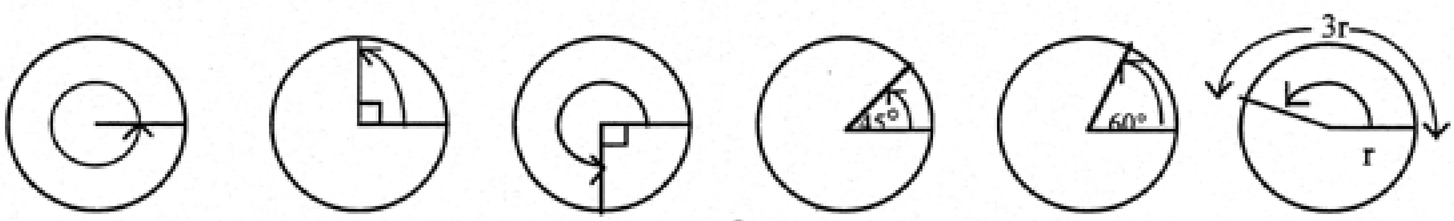
Numerically, we have:

$$1 \text{ rad} = \frac{180°}{\Pi} \approx 57.3°.$$

Now it's your turn:

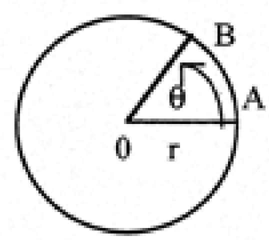
Exercise III.1

Evaluate the following angles in radians :



Remember: θ in radians $= \dfrac{\overset{\frown}{AB}}{r}$,

or $\overset{\frown}{AB} = r\theta$ (with θ in rad).



Thus, for example, on a circle of radius 10cm, an angle of n radians cuts off (intercepts) an arc of length $10n$ cm.

Exercise III.2

Express in radian measure: (a) 30°; (b) 135°; (c) 210°; (d) 315°; (e) 160°; (f) 10°

Exercise III.3

Express in degree measure: (a) $\Pi/3$ rad; (b) $5\Pi/9$ rad; (c) $\Pi/24$ rad; (d) $\Pi/4$ rad; (e) $7\Pi/6$ rad

How about some trig functions of angles expressed in radian measure?:

Exercise III.4

Find: (a) $\sin \Pi/6$; (b) $\cos \Pi/4$; (c) Is it true that $\sin(\Pi/18) = \cos(4\Pi/9)$?

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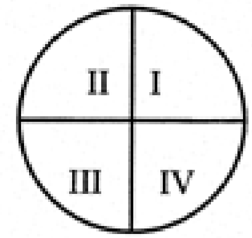
*Note that, if we had used the string to step out straight chords instead of arcs, we would complete the circle with exactly 6 lengths, forming a regular hexagon. You can thus guess that one radian is a bit less than 60°.

You should be familiar with the <u>signs</u> of the trigonometric function for different ranges of angle. This is traditionally done in terms of the four <u>quadrants</u> of the complete circle. It is convenient to make a table showing the signs of the trig functions without regard to their actual values:

	1st. Quad.	2nd. Quad.	3rd. Quad.	4th Quad.
sin (csc)	+	+	-	-
cos (sec)	+	-	-	+
tan (cot)	+	-	+	-



This means that there are always *two* angles between 0 and 2Π that have the same sign and magnitude of any given trig function.

The mnemonic "<u>ALL Students Take Calculus</u>," although it is not a true statement (except at MIT and Cal Tech and maybe a few other places) can remind you which of the main trig functions are **positive** in the successive quadrants.

<u>Exercise III.5</u>

Fill in the following table, giving both signs and magnitudes of the trig functions:

θ	sin θ	cos θ	tan θ	θ	sin θ	cos θ	tan θ
$\Pi/4$				$5\Pi/6$			
$\Pi/3$				$7\Pi/6$			
$2\Pi/3$				$3\Pi/2$			
$3\Pi/4$				$7\Pi/4$			

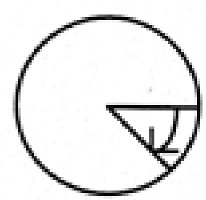
If we are just dealing with triangles, all of the angles are counted positive and are less than 180°(i.e., $< \Pi$). But angles less than 0 rad or more than Π rad are important for describing rotations. For this purpose we use a definite sign convention:

Counterclockwise rotations are positive; clockwise rotations are negative.

Here are some examples:

$+\frac{3\Pi}{2}$ rad

$-\frac{\Pi}{3}$ rad

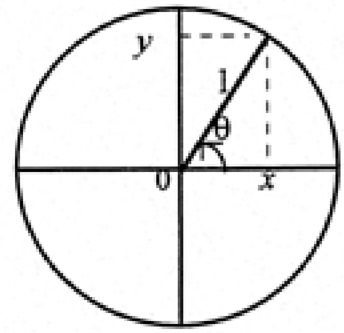


-4Π rad

When we are dealing with rotations, we often talk in terms of numbers of <u>revolutions</u>. Since 1 revolution (rev) is equivalent to 2Π rad, it is easy to convert from one to the other: just <u>multiply revs by 2Π to get rad,</u> or <u>divide rads by 2Π to get revs.</u>

IV. TRIGONOMETRIC FUNCTION AS FUNCTIONS

We have emphasized the usefulness of knowing the values of sin, cos, tan, etc. of various specific angles, but of course the trig functions really *are* functions of a continuous variable -- the angle θ that can have any value between $-\infty$ and $+\infty$. It is important to have a sense of the appearance and properties of the graphs of these functions.



Drawing the unit circle can be of help in visualizing these functions. If we draw a unit circle, and give it x and y axes as shown, then we have:

$$\sin\theta = \frac{y}{1} = y, \cos\theta = \frac{x}{1} = x, \tan\theta = \frac{y}{x}.$$

Notice the following features:

1) Every time we go through an integral multiple of 2Π, we come back to the same point on the unit circle. This means that sin, cos and tan are periodic functions of θ with period 2Π; i.e.,

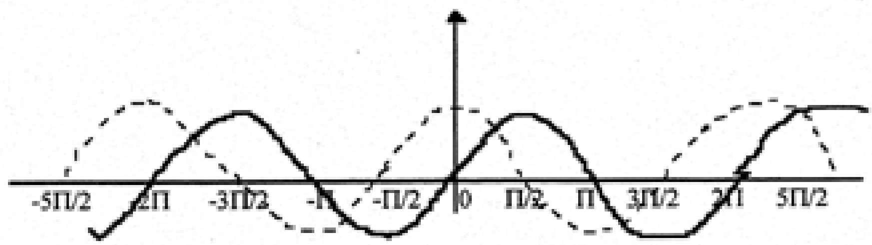
$$\sin(\theta + 2\Pi) = \sin\theta,$$ and similarly for cos and tan.

2) The values of sin and cos never get outside the range between +1 and -1; they oscillate between these limits.

3) cos θ is +1 at $\theta = 0$; sin $\theta = +1$ at $\theta = \Pi/2$. More generally one can put:

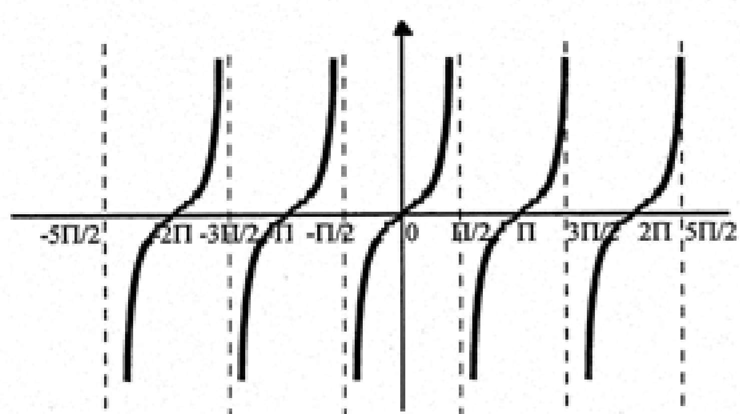
$$\cos\theta = \sin(\theta + \Pi/2), \text{ or } \sin\theta = \cos(\theta - \Pi/2)$$

Thus the whole cosine curve is shifted through Π/2 (negatively) along the θ axis relative to the sine curve, as shown below:



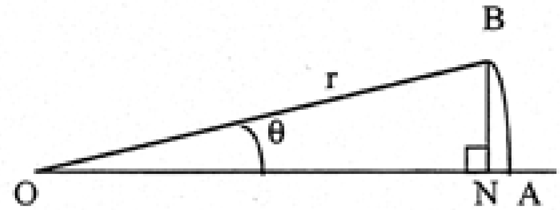
It is easy to construct quite good sketches of these functions with the help of the particular values of sin and cos with which you are familiar.

4) The tangent function becomes infinitely large (+ or -) at odd multiples of Π/2, where the value of x on the unit circle goes to zero. It is made of infinitely many separate pieces, as indicated below:



Like the sine function, the tangent function is zero whenever θ is an integral multiple of Π. Moreover, tan θ is almost equal to sin θ if θ is a *small* angle (say less than about 10° or 0.2rad). For such small angles, both sin and tan are also almost equal to θ itself as measured in radians. This can be seen from the diagram here. We have the following relationships:

$$\theta = \frac{AB}{r} = \frac{AB}{OA} \geq \frac{BN}{OA};$$



$$\sin\theta = \frac{BN}{r} = \frac{BN}{OA;}$$

$$\tan\theta = \frac{BN}{ON} \geq \frac{BN}{OA}.$$

Thus, $$\sin\theta = \theta = \tan\theta.$$

These results are the basis of many useful approximations.

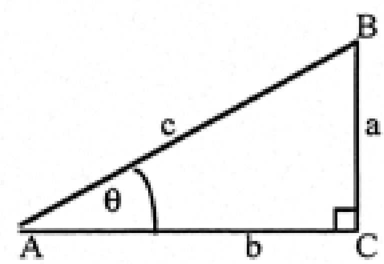
Exercise IV.1

Using your calculator, find sin θ and tan θ for θ = 0.1 rad, 0.15 rad, 0.2 rad, 0.25 rad. (If you do this, you will be able to see that it is always true that sin $\theta = \theta =$ tan θ for these small angles.)

V. TRIGONOMETRIC IDENTITIES

You know that the various trig functions are closely related to one another. For example, cos θ = 1/sec θ. Since this is true for *every* θ, this is an identity, not an equation that can be solved for θ. Trigonometric identities are very useful for simplifying and manipulating many mathematical expressions. You ought to know at least a few of them.

1) Probably the most familiar, and also one of the most useful, is the one based on Pythagoras's Theorem and the definitions of sin and cos:



$$c^2 = a^2 + b^2 \text{, with } \sin\theta = a/c, \cos\theta = b/c;$$

$$\sin^2\theta + \cos^2\theta = 1$$

Dividing through by $\cos^2\theta$ or $\sin^2\theta$ then gives two other identities important in calculus:

$$\tan^2\theta + 1 = \sec^2\theta; \qquad 1 + \cot^2\theta = \csc^2\theta.$$

2) Two other very simple identities are:

$$\sin(-\alpha) = -\sin\alpha; \qquad \cos(-\beta) = \cos\beta.$$

(We say that sin θ is an odd function of θ and cos θ is an even function of θ.)

3) Angle addition formulas: Given any two angles α and β,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta;$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

If you learn these formulas, you can easily construct the formulas for sin or cos of the difference of two angles. (Just use the odd/even properties of sin and cos.)

Exercise V.1.

Use the angle addition formulas to evaluate the following:
(a) $\sin(\theta + 3\Pi/2)$; (b) $\cos(\theta - \Pi/4)$; (c) $\sin(\theta + \Pi/6)$; (d) $\cos(\theta + 7\Pi/4)$.

4) Even if you don't memorize the general results in (3), you should certainly know the formulas obtained when you put $\alpha = \beta = \theta$ -- the double-angle formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta;$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

By using the second of these in reverse, you can develop half-angle formulas:

Exercise V.2

Prove the half-angle formulas :

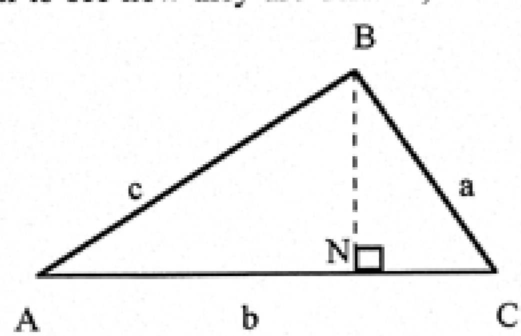
$$\sin\frac{1}{2}\alpha = \pm\sqrt{\frac{1-\cos\alpha}{2}};\ \cos\frac{1}{2}\alpha = \pm\sqrt{\frac{1+\cos\alpha}{2}}.$$

Exercise V.3

Using the results of Exercise V.2, find sin 22.5° and cos 22.5°.

VI. SINE & COSINE LAWS FOR THE GENERAL TRIANGLE

Not everything can be done with right triangles, and you should be familiar with two other sets of identities that apply to a triangle of any shape. Rather than memorizing these forms, you should know how to use them to find angles and side lengths. It's also useful to see how they are derived, namely by dropping a perpendicular and using Pythagoras's Theorem.



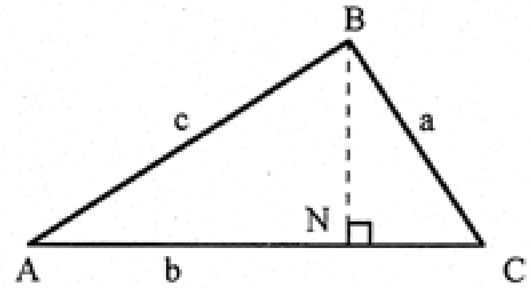
1) The Laws of Sines

For any triangle ABC, labeled as in the diagram:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This result follows from considering the length of a perpendicular drawn from any angle to the opposite side. Such a perpendicular (BN in our diagram) can be calculated in two ways:

$$BN = c \sin A = a \sin C.$$



Rearranging this gives $(\sin A)/a = (\sin C)/c$, and it is pretty obvious that considering either of the other perpendiculars will complete the relationship.

We can use the law of sines to solve a triangle if we know one angle and the length of the side opposite to it, plus one other datum -- either another angle or the length of another side. (We can always make use of the fact that the angles of any triangle add up to 180°.)

Exercise VI.1

In a triangle labeled as in the diagram above, let $A = 30°$, $a = 10$, and $C = 135°$. Find B, b and c.

Exercise VI.2

In another triangle, suppose $B = 50°$, $b = 12$, $c = 15$. Find A, C and a.

2) The Law of Cosines

This is useful if we do not know the values of an angle and its opposite side. What that means, essentially, is that we are given the value of at most one angle. If this is the angle A in the standard diagram, the Law of Cosines states that:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Thus, if b, c, and A are given, we can calculate the length of the third side, a.

The Law of Cosines is kind of extension of Pythagoras's Theorem, and we prove it by using Pythagoras. Take a triangle as labeled before, and again draw a perpendicular from angle B onto AC. Let $BN = h$ and let AN be x, so that $NC = b - x$. Then in $\Delta\, ABN$ we have

$c^2 = x^2 + h^2$, and in $\Delta\, BCN$ we have

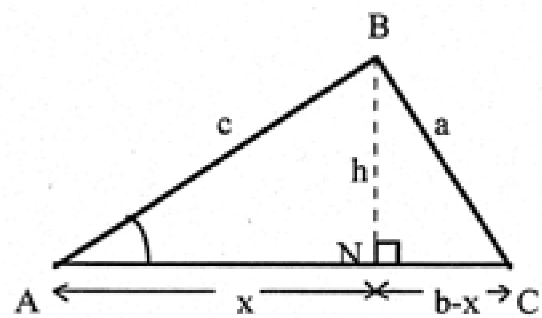
$a^2 = (b\text{-}x)^2 + h^2.$

Combining these gives

$c^2 - a^2 = 2bx - b^2$. But $x = c \cos A$;

substituting this and rearranging then gives

$$a^2 = b^2 + c^2 - 2bc \cos A,$$



as already stated. Doing similar calculations based on drawing perpendiculars from angles A and C gives similar equations for b^2 and c^2 in terms of c, a, B and a, b, C respectively.

(But you don't need to do new calculations; just permute the symbols in the first equation.)

The Law of Cosines also allows us to find the angle A if we are given the lengths of all three sides. For this purpose it can be rewritten as follows:

$$\cos A = \frac{b^2 + c^2 - a^2.}{2bc}$$

As soon as one angle has been determined in this way, we can use the Law of Sines to do the rest.

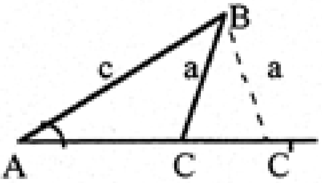
<u>Exercise VI.3</u>

In a triangle labeled as above, $a = 5$, $b = 10$, and $C = 135°$. Find c, A and B.

<u>Exercise VI.4</u>

In another triangle, suppose $a = 10$, $b = 20$, $c = 25$. Find all the angles.
[Draw a reasonably good sketch to see what the triangle looks like, and remember that sin θ and sin $(180° - \theta)$ are equal.]

Warning!: A triangle is completely defined if we know the lengths of two of its sides and the included angle, or any two of its angles and the length of one side. Use of the law of sines or the law of cosines, as appropriate, will give us the rest of the information. However, if we know the lengths of two sides, plus an angle that is <u>not</u> the angle between them, there <u>may</u> be an ambiguity. The diagram here shows an example of this. If we are given the angle A and the sides a and c, there may be two possible solutions, according to whether the angle C is less than 90° or greater than 90°.The magnitude of cos C is defined, but its sign is not. This ambiguity will exist whenever a is less than c--i.e., if the length of the side opposite to the given angle is shorter than the adjacent side.



<u>Exercise VI.5</u>

A triangle has $A = 30°$, $a = 6$, $c = 10$. Find B, C and b.

ANSWERS TO EXERCISES

Note: In a few cases (Exercise II.1, III.5, IV.1, and V.2), answers are not given, because you can so easily check them for yourself.

Exercise I.1. 3/5; 4/5; 3/4; 5/3; 5/4; 4/3.

Exercise I.2. $\sin(90°-\theta) = 5/13$;
$\sin\theta + \cos(90°-\theta) = 12/13 + 12/13 = 24/13$;
$\tan\theta + \cot(90°-\theta) = 12/5 + 12/5 = 24/5$;
$\sec\theta + \csc(90°-\theta) = 13/5 + 13/5 = 26/5$.

Exercise I.3. $\cos A = 24/25$; $\tan A = 7/24$; $\csc A = 25/7$; $\sec A = 25/24$; $\cot A = 24/7$.

Exercise I.4. (a) $5 \sin 20° = 1.71$; (b) $8 \tan 40° = 6.71$; (c) $6 \sec 53° = 9.97 \approx 10$ -- a 3:4:5 triangle. (More precisely, the angles in such a triangle are 36.9° and 53.1°.)

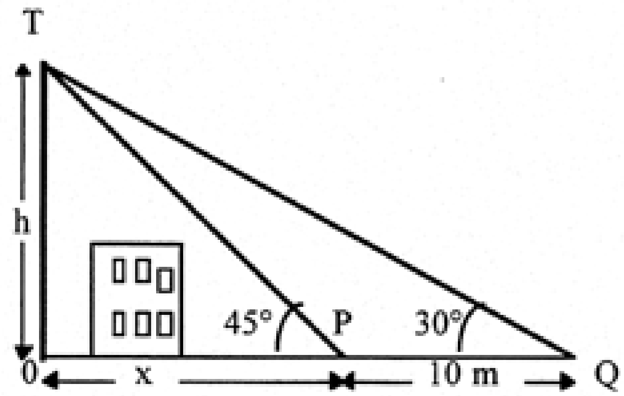
Exercise II.1. You should be able to check this for yourself, using your calculator.

Exercise II.2.

	A	B	C	a	b	c
(a)	30°	60°	90°	$2\sqrt{3}$	6	$4\sqrt{3}$
(b)	45°	45°	90°	13	13	$13\sqrt{2}$
(c)	60°	30°	90°	$5\sqrt{3}$	5	10
(d)	45°	45°	90°	$6\sqrt{2}$	$6\sqrt{2}$	12
(e) Ha! Can't be! $\sqrt{2} \approx 1.41 > 1$. Requires Leg > Hypotense.						
(f)	0°	90°	90°	0	4	4

Exercise II.3

Let the height of the top of the pole above eye level be h, and let the unknown distance OP be x.
We can make double use of the relation $a = b \tan\theta$.
In the triangle OPT, we have $h = OP \tan 45° = x \tan 45° = x$.



In the triangle OQT, we have $h = OQ \tan 30° = (x + 10) \tan 30° = (x + 10)(1/\sqrt{3})$.
Multiplying the second equation throughout by $\sqrt{3}$ gives:

$$\sqrt{3}\,h = x + 10.$$

Substituting $x = h$ from the first equation gives $\sqrt{3} = h + 10$, and so $h = 10/(\sqrt{3}-1)$.
Putting $\sqrt{3} \approx 1.73$ gives $h \approx 10/(0.73) = 13.7$m.
[Alternatively, we could have put $x = h \cot 45°$, $x + 10 = h \cot 30°$, and eliminated x by

subtraction right away. Here we've used angles for which you know the values of the trig functions. But you could solve any similar problem with arbitrary angles. Suppose we put $OPT = \alpha$, $OQT = \beta$, $PQ = d$. Then you could put:

$h = x \tan \alpha$, or $x = h \cot \alpha$: $h = (x + d) \tan \beta$, or $x + d = h \cot \beta$.

Using the cotangents is more direct. You can verify that the result is $h = d/(\cot \alpha - \cot \beta)$. **Or**, you could have tackled this particular problem (with its angles of 45° and 30°) in a different way, by using the known ratios of the sides in the triangles *OPT* and *OQT*. Do this for the experience! (But of course this cannot be used as a general method.)]

Exercise III.1. 2Π; Π/2; 3Π/2; Π/4 Π/3; 3.

Exercise III.2. (a) Π/6; (b) 3Π/4; (c) 7Π/6; (d) 7Π/4; (e) 8Π/9; (f) Π/18.

Exercise III.3. (a) 60°; (b) 100°; (c) 7.5°; (d) 45°; (e) 210°.

Exercise III.4. (a) sin (Π/6) = sin 30° = 1/2; (b) cos (Π/4) = cos 45° = $1/\sqrt{2}$; (c) We won't take "no" for an answer!--sin (Π/18) = cos [90° - (Π/18)] = cos [(Π/2) - (Π/18)] = cos (8Π/18) = cos (4Π/9).

Exercise III.5. You should be able to check these results for yourself.

Exercise IV.1. You will already have checked these results.

Exercise V.1. (a) $-\cos\theta$; (b) $(1/\sqrt{2})(\cos\theta + \sin\theta)$; (c) $(\sqrt{3}/2)\sin\theta + (1/2)\cos\theta$; (d) $(1/\sqrt{2})(\cos\theta + \sin\theta)$. [Note that the results of (b) and (d) are the same, because (θ - Π/4) and (θ + 7Π/4) are separated by 2Π.]

Exercise V.2. You should have no trouble obtaining the stated results from the preceding formulas; it's just algebra. Think about the ambiguities of sign, though.

Exercise V.3. 0.38; 0.92. (Get more significant digits if you like, and check with your calculator.)

Exercise VI.1. $B = 15°$; $b = 5.2$; $c = 14.1$.

Exercise VI.2. $A = 56.8°$, $C = 73.2°$, $a = 13.1$.

Exercise VI.3. $c = 14.0$, $A = 14.6°$, $B = 30.3°$.

Exercise VI.4. $A = 22.3°$, $B = 49.5°$, $C = 108.2°$.

Exercise VI.5. $\sin C = 5/6$, permitting $C = 56.4°$, $B = 93.6°$, $b = 12$, or $C = 123.6°$, $B = 26.4°$, $b = 5.3$.

This module is based in large part on an earlier module prepared by the Department of Mathematics.